ON THE REPRESENTATION DIMENSION OF SCHUR ALGEBRAS

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Abstract. Lower bounds for the representation dimension of Schur algebras for GL_n in characteristic $p \geq 2n-1$ are established. In particular it is shown that for fixed n the representation dimensions of the Schur algebras get arbitrarily large.

1. Introduction

The representation dimension of an artin algebra is an invariant introduced by Auslander [2]. He has shown that an artin algebra has representation dimension at most two if and only if it has finite representation type. This lead him to the expectation that for representation infinite algebras the representation dimension measures how far the algebra is from having finite representation type.

Recent results [1, 4, 5, 15, 14] suggest that the representation dimension measures how complicated homologically the representation theory of an artin algebra is.

In this paper we study the representation dimension of Schur algebras $S(n, r)$ for GL_n, which form one of the most widely studied classes of quasi-hereditary algebras. We construct a lower bound for the representation dimension of these algebras and show that, as r increases, this lower bound gets arbitrarily large. Our strategy towards that goal is the following:

We first show that, in order to establish a lower bound for the representation dimension of an artin algebra Λ, it suffices to find a finite dimensional complete intersection as the endomorphism ring of some projective Λ-module (see Corollary 3.3).

The main part of the paper is then devoted to showing that over Schur algebras there is a projective module with such an endomorphism ring. This is achieved by constructing small projective injective modules with endomorphism ring $\mathbb{k}[x]/(x^m)$ for some m, and tensoring up their Frobenius twists while assuring they remain projective injective.

2. Representation dimension

In this section we recall the definition of representation dimension and some of its properties. We then give the definition of the dimension of a triangulated category and of the derived dimension of an exact category, and the inequalities between all these dimensions which will be used to establish lower bounds for the representation dimension.

Throughout this paper we assume all algebras to be artin algebras over a commutative artin ring $\mathbb{k}$. For an algebra Λ we denote by Λ-mod the category of finitely generated left Λ-modules, and by $D^b(\Lambda\text{-mod})$ its bounded derived category.

2.1. **Definition** (Auslander [2])**.** Let Λ be an artin algebra. Then the *representation dimension* of Λ is

$$\operatorname{repdim} \Lambda = \min\{\operatorname{gldim} \operatorname{End}_\Lambda(M) \colon M \in \Lambda\text{-mod a generator and cogenerator}\}.$$

Here M being a generator and cogenerator means that all indecomposable projective and all indecomposable injective Λ-modules are isomorphic to direct summands of M.

This definition is motivated by the following fact.

2.2. **Theorem** (Auslander [2]). *Let Λ be an artin algebra. Then* $\operatorname{repdim} \Lambda \leq 2$ *if and only if Λ is of finite representation type (that is, has only finitely many indecomposable modules up to isomorphism).*

This result suggests that for representation infinite algebras the representation dimension should measure (in some sense) how far the algebra is from having finite representation type. The following two results show that the representation dimension is a reasonable invariant:

2.3. **Theorem** (Iyama [11]). *Let Λ be an artin algebra. Then* $\operatorname{repdim} \Lambda < \infty$.

2.4. **Theorem** (Rouquier [16]). *For any $n \in \mathbb{N}^{\geq 2}$ there is an artin algebra Λ with* $\operatorname{repdim} \Lambda = n$.

Derived dimension. The following construction of is due to Rouquier [16, 17]:

Let $\mathcal{T}$ be a triangulated category, and $\mathcal{I}$, $\mathcal{J} \subseteq \mathcal{T}$ (we may think of them as subcategories or just as subclasses of objects). Then we set

$$\begin{aligned}
\langle \mathcal{I} \rangle &= \operatorname{add}\{I[i] \colon i \in \mathbb{Z}\}, \\
\mathcal{I} * \mathcal{J} &= \{X \in \mathcal{T} \colon \text{there is an exact triangle } I \longrightarrow X \longrightarrow J \longrightarrow I[1] \\
&\qquad \text{with } I \in \mathcal{I}, J \in \mathcal{J}\}, \\
\mathcal{I} \diamond \mathcal{J} &= \langle \mathcal{I} * \mathcal{J} \rangle, \\
\langle \mathcal{I} \rangle_1 &= \langle \mathcal{I} \rangle, \text{ and} \\
\langle \mathcal{I} \rangle_{n+1} &= \langle \mathcal{I} \rangle_n \diamond \langle \mathcal{I} \rangle \text{ for } n \geq 1.
\end{aligned}$$

Moreover if $\mathcal{I} = \{T\}$ for some $T \in \mathcal{T}$ we write $\langle T \rangle_n$ instead of $\langle \{T\} \rangle_n$.

2.5. **Definition.** (1) Let $\mathcal{T}$ be a triangulated category. Then the *dimension of* $\mathcal{T}$ (introduced in [17]) is

$$\dim \mathcal{T} = \inf\{n \in \mathbb{N}_0 \colon \exists T \in \mathcal{T} \text{ such that } \langle T \rangle_{n+1} = \mathcal{T}\}.$$

(2) Let $\mathcal{E}$ be an exact category. Then the *derived dimension of* $\mathcal{E}$ (see [14]) is

$$\begin{aligned}
\operatorname{derdim} \mathcal{E} = \inf\{n \in \mathbb{N}_0 \colon \exists T \in D^b(\mathcal{E}) \text{ such that } \mathcal{E} \subseteq \langle T \rangle_{n+1} \\
\text{as subcategories of } D^b(\mathcal{E})\}.
\end{aligned}$$

We will need the following relations between dimension, derived dimension and representation dimension.

2.6. **Lemma.** *Let Λ be an artin algebra, which is not semisimple. Then*

$$\operatorname{repdim} \Lambda \geq \operatorname{derdim} \Lambda\text{-mod} + 2.$$

If moreover Λ is self-injective then the stable module category Λ-$\underline{\text{mod}}$ is a triangulated category, and

$$\operatorname{derdim} \Lambda\text{-mod} \geq \dim \Lambda\text{-}\underline{\text{mod}}.$$

Proof. See [14, Section 1]. $\square$

3. Endomorphism rings of projective modules

3.1. **Proposition.** *Let Λ be an artin algebra, P a projective Λ-module, and $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ (or, equivalently, let e be an idempotent in Λ and $\Gamma = e\Lambda e$). Then*

$$\operatorname{derdim} \Lambda\text{-mod} \geq \operatorname{derdim} \Gamma\text{-mod},$$

and

$$\dim D^b(\Lambda\text{-mod}) \geq \dim D^b(\Gamma\text{-mod}).$$

Proof. The functor $\operatorname{Hom}_\Lambda(P, -)\colon \Lambda\text{-mod} \longrightarrow \Gamma\text{-mod}$ is exact, hence induces a functor between the corresponding derived categories.

We first prove the first inequality. We have a natural transformation ϕ defined by

$$\begin{aligned} \phi\colon 1_{\Gamma\text{-mod}} &\longrightarrow \operatorname{Hom}_\Lambda(P, P \otimes_\Gamma -) \\ \phi_N\colon N &\longrightarrow \operatorname{Hom}_\Lambda(P, P \otimes_\Gamma N) \\ n &\longmapsto [p \longmapsto p \otimes n]. \end{aligned}$$

Since ϕ_Γ is an isomorphism and both functors are right exact ϕ is a natural isomorphism, and hence the functor $\operatorname{Hom}_\Lambda(P, -)$ maps Λ-mod densely to Γ-mod.

Now assume $\operatorname{derdim} \Lambda\text{-mod} = n < \infty$ (otherwise there is nothing to show). Then there is T in $D^b(\Lambda\text{-mod})$ such that any $\Lambda\text{-mod} \subseteq \langle T \rangle_{n+1}$. Let $X \in \Gamma$-mod. Since $\operatorname{Hom}_\Lambda(P, -)$ is dense we have $X = \operatorname{Hom}_\Lambda(P, Y)$ for some $Y \in \Lambda$-mod. Since $Y \in \langle T \rangle_{n+1}$ and the functor $\operatorname{Hom}_\Lambda(P, -)$ is exact it follows that $X = \operatorname{Hom}_\Lambda(P, Y) \in \langle \operatorname{Hom}_\Lambda(P, T) \rangle_{n+1}$.

For the proof of the second inequality let $X \in D^b(\Gamma\text{-mod})$. We may think of X as a bounded complex of Γ-modules. Applying $P \otimes_\Gamma -$ to this complex degree wise, we obtain a bounded complex Y of Λ-modules. Note that applying $P \otimes_\Gamma -$ degree wise does not induce a functor between the derived categories. Nevertheless, the complex X is mapped to Y by the functor $\operatorname{Hom}_\Lambda(P, -)$, and hence this functor is also dense as a functor between the derived categories. The inequality now follows as above. □

3.2. **Corollary.** *Let Λ be an artin algebra, P a projective Λ-module, and assume $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ is selfinjective. Then*

$$\operatorname{repdim} \Lambda \geq \operatorname{derdim} \Lambda\text{-mod} + 2 \geq \operatorname{derdim} \Gamma\text{-mod} + 2 \geq \dim \Gamma\text{-}\underline{\mathrm{mod}} + 2.$$

Proof. This is just Proposition 3.1 and Lemma 2.6. □

3.3. **Corollary.** *Let Λ be a finite dimensional algebra over a field $\Bbbk$, P a projective Λ-module, and assume $\operatorname{End}_\Lambda(P)$ is a complete intersection of codimension n (that is a commutative algebra of the form $\Bbbk[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$ for some natural numbers $a_1, \ldots, a_n \geq 2$). Then*

$$\operatorname{repdim} \Lambda \geq n + 1.$$

Proof. By [4], Theorem 3.2 and the proof of 3.5, we know that $\dim \operatorname{End}_\Lambda(P)^{\mathrm{op}}\text{-}\underline{\mathrm{mod}} \geq n - 1$. The claim now follows from our Corollary 3.2. □

4. Applications to GL_n

Background and Notation. Throughout the following, let $G = \mathrm{GL}_n(\Bbbk)$ for an algebraically closed field $\Bbbk$ of characteristic $p \geq 2n-1$. This assumption is necessary for Lemma 4.5 to hold, though Donkin conjectures [8, Conjecture (2.2)] this holds for smaller p as well. However, some of our combinatorial argumants require at least $p > n$. It is well-known that the category of polynomial G-representations of degree r is equivalent to the category $S(n, r)$-mod, where $S(n, r) = \operatorname{End}_{\Bbbk\Sigma_r}(V^{\otimes r})$

is called a Schur algebra. Here $V = \Bbbk^n$ is the natural G-module, and Σ_r is the symmetric group on r letters, which acts on $V^{\otimes r}$ by permutation of tensors. We do not distinguish between an $S(n,r)$-module and the corresponding G-module. A partition $\lambda \vdash r$ is a weakly decreasing sequence of natural numbers $(\lambda_1, \lambda_2, \dots)$ such that $\sum_{i\geq 1} \lambda_i = r$. A partition λ is said to have at most n parts, if $\lambda_i = 0$ for all $i > n$. Set $\Lambda(n,r) = \{\lambda \vdash r | \lambda \text{ has at most } n \text{ parts}\}$ and $\Lambda(n) := \bigcup_{r\geq 1} \Lambda(n,r)$. Irreducible $S(n,r)$-modules are indexed by partitions $\lambda \in \Lambda(n,r)$. The algebra $S(n,r)$ is quasi-hereditary with respect to the dominance order on $\Lambda(n,r)$ where $\lambda \leq \mu$ if $\sum_{1\leq i\leq k} \lambda_i \leq \sum_{1\leq i\leq k} \mu_i$ for all k. We will freely use the theory of quasi-hereditary algebras. For a good account of this, see e.g. [7]. We use the usual notation: $P_n(\lambda)$ for the projective, $T_n(\lambda)$ for the direct summand of the characteristic tilting module, $\Delta_n(\lambda)$ for the standard module and $L_n(\lambda)$ for the simple module corresponding to $\lambda \in \Lambda(n)$. If no index is given on P, T, Δ, the index is assumed to be n. We will generally omit indices on L, as the underlying group will always be clear from the context. We define addition and subtraction of partitions componentwise. More generally on can define compositions $(\lambda_1, \dots, \lambda_n)$ of r where $\sum_{1\leq i\leq n} \lambda_i = n$ but the sequence is not necessarily weakly decreasing. In particular we define the compositions $\epsilon^i = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in position i. Let $\omega = \omega^n = \epsilon^1 + \cdots + \epsilon^n = (1^n)$ which corresponds to the determinant representation $L(\omega)$. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda(n,r)$ we denote by $w_0\lambda$ the composition $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$. For an account of abacus notation of partitions and p-cores see e.g. [18]. A partition λ is called p-regular if $\lambda_i = \lambda_{i+k}$ implies $k < p$. A partition is called column p-regular if $\lambda_i - \lambda_{i+1} < p$ for all $i \in \mathbb{N}$. Let Mull denote the Mullineux involution on p-regular partitions. For more detail about this, see e.g. [12].

Some facts about Schur algebras:

(1) Let $\{\varepsilon_\lambda \mid \lambda \in \Lambda(n,r)\}$ be a set of orthogonal idempotents such that $S(n,r)\varepsilon_\lambda \cong P(\lambda)^{\oplus \dim L(\lambda)}$. Let
$$f_{k,l} = \sum_{\sum_{1\leq i\leq k} \lambda_i \geq l} \varepsilon_\lambda.$$
Then, since the set of all partitions λ with $\sum_{1\leq i\leq k} \lambda_i \geq l$ is saturated in the dominance order, the general theory of quasi-hereditary algebras tells us that $S(n,r)_{\geq_k l} := f_{k,l} S(n,r) f_{k,l}$ is a quasi-hereditary subalgebra of $S(n,r)$ and $S(n,r)f_{k,l}S(n,r)$ is a heredity ideal, giving rise to a quasi-hereditary quotient
$$S(n,r)_{\leq_k l} := S(n,r)/S(n,r)f_{k,l+1}S(n,r).$$
Denote by $S(n,r)_{=_k l}$ the subquotient
$$S(n,r)_{=_k l} := f_{k,l}S(n,r)f_{k,l}/f_{k,l}S(n,r)f_{k,l+1}S(n,r)f_{k,l}$$
which is again quasi-hereditary.

(2) By [10, Theorem 6.3 and Section 8], there exists and isomorphism
$$S(n,r)_{=_1 l} \cong S(n-1,r)_{\leq_1 l},$$
inducing an equivalence of categories
$$S(n,r)_{=_1 l}\text{-mod} \to S(n-1,r)_{\leq_1 l}\text{-mod},$$
which sends a simple module $L(\lambda_1, \dots, \lambda_n)$ (with $\lambda_1 = l$) to $L(\lambda_2, \dots, \lambda_n)$ and respects the quasi-hereditary structure.

(3) For $r \geq n$ there exists an epimorphism $S(n,r) \twoheadrightarrow S(n, r-n)$. This yields an isomorphism $S(n, r-n) \cong S(n,r)_{\leq_{n-1} r-1}$, again inducing an equivalence of module categories in which $L(\nu)$ is sent to $L(\nu + \omega^n)$ and corresponds

to tensoring with the determinant representation. The quasi-hereditary structure is respected. The epimorphism $S(n,r) \twoheadrightarrow S(n,r-n)$ also yields a fully faithful exact functor $\mathcal{J}_n \colon S(n,r-n)$-mod $\to S(n,r)$-mod, again sending a simple $L(\nu)$ to $L(\nu+\omega^n)$ and a standard module $\Delta(\nu)$ to $\Delta(\nu+\omega^n)$.

(4) Furthermore, for $e_{n-1} := f_{n-1,r}$, we have $e_{n-1}S(n,r)e_{n-1} \cong S(n-1,r)$ and hence an exact functor $\mathcal{R}_{n-1} = e_{n-1}S(n,r) \otimes_{S(n,r)} - \colon S(n,r)$-mod $\to S(n-1,r)$-mod.

Let $\delta = \delta^n$ be the partition $(n-1, n-2, \ldots, 1, 0)$ of $\frac{n(n-1)}{2}$. We use the usual notation for the root system of type A_{n-1}, denoting the simple roots by α_i (for $1 \le i \le n-1$) and the non-degenerate bilinear form between the root lattice and the corresponding weight lattice (identified with $\Lambda(n)$) by $(-,-)$. Let X_m be the set of all partitions $\lambda \in \Lambda(n)$ such that $(\lambda, \alpha_j) \le p^m - 1$ for all simple roots α_j. This is the same as saying that $\lambda_j - \lambda_{j+1} \le p^m - 1$ for all $1 \le j \le n-1$. Let $-^F$ denote the Frobenius twist.

4.1. **Theorem** (Steinberg's tensor product theorem)**.** *Every partition can be written uniquely as a p-adic sum over column p-regular partitions, say $\tau = \sum_{0 \le i \le m} p^i \tau_i$. Then*
$$L(\tau) \cong \bigotimes_{0 \le i \le m} L(\tau_i)^{F^i}.$$

4.2. **Remark.** Steinberg's tensor product theorem says that for any set $\{\alpha_i \mid \alpha_i \in X_1, 0 \le i \le m\}$,
$$L(\sum_{0 \le i \le m} p^i \alpha_i) \cong \bigotimes_{0 \le i \le m} L(\alpha_i)^{F^i}.$$

However, in this situation, the expression is not unique for GL_n. E.g. the p-fold tensor product of the determinant module with itself is the simple $L(p, \ldots, p)$, which satisfies $(p, \ldots, p) \in X_1$, so $L(p, \ldots, p) \cong L(p, \ldots, p)^{F^0}$, but it is also isomorphic to $L(1, \ldots, 1)^F$. Choosing column p-regular partitions gets rid of this ambiguity.

Let $d(\lambda) = \max\{d \ge 0 | \lambda_i - \lambda_{i+1} \equiv -1 \bmod p^d \text{ for all } 1 \le i < n\}$. The following lemma is taken from [9, Section 1,(7)].

4.3. **Lemma.** *For two partitions λ and μ the following are equivalent:*

(a) *λ and μ belong to the same G-block;*

(b) *$d(\lambda) = d(\mu) =: d$ and there exist a number a and partitions χ and ψ with the same p-core such that we can write λ and μ in the form $\lambda = (p^d - 1)\delta + a\omega + p^d\chi$, $\mu = (p^d-1)\delta + a\omega + p^d\psi$.*

We will also use the following from [6, Lemma 3.3].

4.4. **Lemma.** *Let $\lambda \in \Lambda(n)$ be column p-regular. Then $\mathrm{Mull}(\lambda')$ has at most n parts and we have $P(\lambda) \cong T(\mathrm{Mull}(\lambda'))$.*

4.5. **Lemma.** *For λ^i column p-regular, $(0 \le i \le m)$, we have*
$$\bigotimes_{0 \le i \le m} P(\lambda^i)^{F^i} \cong \bigotimes_{0 \le i \le m} T(\mathrm{Mull}((\lambda^i)'))^{F^i} \cong T(\sum_{0 \le i \le m} p^i\, \mathrm{Mull}((\lambda^i)'))$$
and this module is a quotient of $P(\sum_{0 \le i \le m} p^i \lambda^i)$.

Proof. For λ^i column p-regular, we know by Lemma 4.4 that $P(\lambda^i) \cong T(\mathrm{Mull}((\lambda^i)'))$. Hence
$$\bigotimes_{0 \le i \le m} P(\lambda^i)^{F^i} \cong \bigotimes_{0 \le i \le m} T(\mathrm{Mull}((\lambda^i)'))^{F^i}.$$

Use induction on m, [8, Proposition (2.1)] and [8, Section 2, Example 1] to see that this is isomorphic to the indecomposable tilting module $T(\sum_{0\leq i\leq m} p^i \operatorname{Mull}((\lambda^i)'))$ and has a simple top. For $m=0$ there is nothing to show. Assume therefore that $\bigotimes_{0\leq i\leq m-1} T(\operatorname{Mull}((\lambda^i)'))^{F^i}$ is indecomposable, isomorphic to

$$T(\sum_{0\leq i\leq m-1} p^i \operatorname{Mull}((\lambda^i)'))$$

and has a simple top. Then [8, Proposition (2.1)] tells us that

$$(\bigotimes_{0\leq i\leq m-1} T(\operatorname{Mull}((\lambda^i)'))^{F^i}) \otimes T(\operatorname{Mull}((\lambda^m)'))^{F^m}$$

is again a tilting module. As in [8, Section 2, Example 1] (by our assumption that $p \geq 2n-1$), since $\sum_{0\leq i\leq m-1} p^i \operatorname{Mull}((\lambda^i)') \in X_m$, it follows that the restriction of $T(\sum_{0\leq i\leq m-1} p^i \operatorname{Mull}((\lambda^i)'))$ to the mth Frobenius kernel is an indecomposable projective. Under this condition, [8, Proposition (2.1)] asserts that $(\bigotimes_{0\leq i\leq m-1} T(\operatorname{Mull}((\lambda^i)'))^{F^i}) \otimes T(\operatorname{Mull}((\lambda^m)'))^{F^m}$ is also indecomposable and isomorphic to $T(\sum_{0\leq i\leq m} p^i \operatorname{Mull}((\lambda^i)'))$. Applying the argument from [8, Section 2, Example 1] again to $\sum_{0\leq i\leq m} p^i \operatorname{Mull}((\lambda^i)') \in X_{m+1}$, we see that the restriction of this to the $m+1$st Frobenius kernel is again a projective indecomposable, hence $T(\sum_{0\leq i\leq m} p^i \operatorname{Mull}((\lambda^i)'))$ has a simple top. This simple top, by the first of our isomorphisms above, is $L(\sum_{0\leq i\leq m} p^i\lambda^i)$. Therefore,

$$\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i} \cong \bigotimes_{0\leq i\leq m} T(\operatorname{Mull}((\lambda^i)'))^{F^i} \cong T(\sum_{0\leq i\leq m} p^i \operatorname{Mull}((\lambda^i)'))$$

is a quotient of $P(\sum_{0\leq i\leq m} p^i\lambda^i)$ as stated. □

4.6. **Lemma.** *For λ^i, $(0\leq i\leq m)$, column p-regular,*

$$\operatorname{End}_G(\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i}) \cong \bigotimes_{0\leq i\leq m} \operatorname{End}_G(P(\lambda^i)).$$

Proof. We have isomorphisms

$$\begin{aligned}\bigotimes_{0\leq i\leq m} \operatorname{End}_G(P(\lambda^i)) &\cong \bigotimes_{0\leq i\leq m} \operatorname{End}_G(P(\lambda^i)^{F^i})\\ &\cong \operatorname{End}_{G\times\cdots\times G}(\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i})\end{aligned}$$

and an embedding

$$\operatorname{End}_{G\times\cdots\times G}(\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i}) \hookrightarrow \operatorname{End}_G(\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i}).$$

By the previous lemma, $\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i}$ is a quotient of the indecomposable projective $P(\sum_{0\leq i\leq m} p^i\lambda^i)$. As such, the dimension of its endomorphism ring is bounded above by the number of times the top composition factor appears in the module.

Now $\bigotimes_{0\leq i\leq m} P(\lambda^i)^{F^i}$ has a filtration with subquotients $\bigotimes_{0\leq i\leq m} L(\alpha^i)^{F^i}$ for composition factors $L(\alpha^i)$ in $P(\lambda^i)$.

We claim that no filtration subquotient $\bigotimes_{0\leq i\leq m} L(\alpha^i)^{F^i}$ can yield composition factors isomorphic to $L(\sum_{0\leq i\leq m} p^i\lambda^i)$ unless $\alpha^i=\lambda^i$ for all i. To see this, assume $L(\sum_{0\leq i\leq m} p^i\lambda^i)$ is a composition factor of $\bigotimes_{0\leq i\leq m} L(\alpha^i)^{F^i}$. Write α^i in its p-adic expansion $\alpha^i=\alpha^{i,0}+p\alpha^{i,1}+\cdots+p^l\alpha^{i,l}$ with $\alpha^{i,j}$ column p-regular for all j (where l is large and $\alpha^{i,j}=\emptyset$ for $j\gg 0$).

Then

$$\begin{aligned}
\bigotimes_{0\le i\le m} L(\alpha^i)^{F^i} &\cong \bigotimes_{0\le i\le m}\Big(\bigotimes_{0\le j\le l} L(\alpha^{i,j})^{F^j}\Big)^{F^i}\\
&\cong \bigotimes_{0\le i\le m}\bigotimes_{0\le j\le l} L(\alpha^{i,j})^{F^{i+j}}\\
&\cong \bigotimes_{0\le k\le m+l}\bigotimes_{0\le j\le k} L(\alpha^{k-j,j})^{F^k}
\end{aligned}$$

The terms with Frobenius twists of higher order than k cannot contribute terms of degree k in the p-adic expansion of a composition factor of this module, i.e. for a composition factor $L(\tau = \tau^0 + p\tau^1 + \cdots + p^s\tau^s)$ we have that $\tau^0 = \alpha^{0,0}$, $\tau^1 = \beta^0$ for a composition factor $L(\beta)$ of $L(\alpha^{0,1})\otimes L(\alpha^{1,0})$, $\tau^2 = \gamma^0$ for a composition factor $L(\gamma)$ of $L(\alpha^{0,2})\otimes L(\alpha^{1,1})\otimes L(\alpha^{2,0})\otimes L(\beta_1)$, etc.

For $\tau = \lambda$ we see that necessarily $\alpha^{0,0} = \lambda^0$. As α^0 and λ^0 are partitions of the same number, this forces $\alpha^0 = \lambda^0$ and $\alpha^{0,j} = \emptyset$ for $j > 0$. Then $\lambda^1 = \beta^0$ for a composition factor $L(\beta)$ of $L(\emptyset)\otimes L(\alpha^{1,0}) \cong L(\alpha^{1,0})$, hence $\beta = \beta^0 = \alpha^{1,0}$. Again, by α^1 and λ^1 being partitions of the same number, we obtain $\alpha^1 = \alpha^{1,0} = \lambda^1$.

By using induction over i, we see that $\lambda^i = \alpha^{i,0}$ and hence $\lambda^i = \alpha^i$ for all i.

Therefore the dimension of $\mathrm{End}_G(\bigotimes_{0\le i\le m} P(\lambda^i)^{F^i})$ is bounded above by the product of the dimensions of the individual $\mathrm{End}_G(P(\lambda^i))$, hence the inclusion in the above embedding is an isomorphism. □

4.7. **Lemma.** *For $\lambda \in \Lambda(n)$, with $\lambda_1 < p$, the partition $(p-1)\delta + w_0\lambda$ is column p-regular and* $\mathrm{Mull}(((p-1)\delta + w_0\lambda)') = (p-1)\delta + \lambda$.

Proof. For a simple root α_j

$$\begin{aligned}
((p-1)\delta + w_0\lambda, \alpha_j) &= ((p-1)\delta, \alpha_j) + (w_0\lambda, \alpha_j)\\
&= p - 1 - (\lambda, \alpha_{n-j}).
\end{aligned}$$

Since the first part of λ is less than or equal to $p-1$, we have $0 \le (\lambda, \alpha_{n-j}) \le p-1$, hence $0 \le ((p-1)\delta + w_0\lambda, \alpha_j) \le p-1$ and $(p-1)\delta + w_0\lambda \in X_1$, as needed. Its $n-$th component is $\lambda_1 \le p-1$, so it is in fact column p-regular. Lemma 4.4 then asserts that $P((p-1)\delta + w_0\lambda)$ is a tilting module and isomorphic to $T(\mathrm{Mull}(((p-1)\delta+w_0\lambda)'))$. It now follows from [6, Theorem 4.1.(ii)] that $\mathrm{Mull}(((p-1)\delta + w_0\lambda)') = (p-1)\delta + \lambda$. □

4.8. **Lemma.** *Let $\lambda^i \in \Lambda(n)$, $0 \le i \le m-1$, with $\lambda_1^i < p$. Set $\mu := (p^m-1)\delta + \sum_{0\le i\le m-1} p^i w_0\lambda^i$ and $\tilde\mu := (p^m-1)\delta + \sum_{0\le i\le m-1} p^i\lambda^i$. Let γ be column p-regular with $\gamma_1 < (n-1)(p-1)$.*

Then we have the following isomorphisms:

$$\begin{aligned}
\Big(\bigotimes_{i=0}^{m-1} P((p-1)\delta + w_0\lambda^i)^{F^i}\Big)\otimes P(\gamma)^{F^m} &\cong \Big(\bigotimes_{i=0}^{m-1} T((p-1)\delta + \lambda^i)^{F^i}\Big)\otimes T(\mathrm{Mull}(\gamma'))^{F^m}\\
&\cong T(\tilde\mu + p^m\,\mathrm{Mull}(\gamma'))\\
&\cong P(\mu + p^m\gamma)
\end{aligned}$$

Proof. The first three isomorphisms follow from Lemma 4.7 and Lemma 4.5.

Now according to the proof of [6, Theorem 5.1] $T((p^m-1)\delta + \lambda + p^m\,\mathrm{Mull}(\gamma'))$ with $\lambda_1 < p^m$ and γ as in our statement is projective injective and isomorphic to $P((p^m-1)\delta + w_0\lambda + p^m\gamma)$ if and only if $\bar T(\gamma)$ is injective, where the $\bar T(\gamma)$ denotes the tilting module for a certain quantum group $\bar G$ which is isomorphic to G. But $\lambda_1 < p^m$ is satisfied for $\lambda = \sum_{0\le i\le m-1} p^i\lambda^i$ with $\lambda_1^i < p$ for every i.

Now $\bar{T}(\mathrm{Mull}(\gamma'))$ is projective injective as a polynomial $\bar{G}$-module if and only if $T(\mathrm{Mull}(\gamma'))$ is projective injective as a polynomial G-module by the isomorphism between G and $\bar{G}$, and we know by Lemma 4.4 that this is the case for our chosen γ. Therefore, we obtain the third isomorphism. Hence all four the modules are isomorphic. □

4.9. **Remark.** The result of [6, Theorem 5.1] relies on a certain conjecture, but the statement we use from its proof does not.

4.10. **Lemma.** *Any $r \in \mathbb{N}$ can be written uniquely as $r = \sum_{i=0}^{m} u_i p^i$ with $(p-1)|\delta| + 1 \leq u_i \leq (p-1)|\delta| + p$ for $0 \leq i \leq m-1$ and $0 \leq u_m \leq (p-1)|\delta|$.*

Proof. We proceed by induction on r. The statement is clear for $r \leq (p-1)|\delta|$. For $r > (p-1)|\delta|$ write $r = bp + j$ where $0 \leq j \leq p-1$. Then find k such that $pk + j$ satisfies $(p-1)|\delta| + 1 \leq pk + j \leq (p-1)|\delta| + p$ which is possible since the interval has size p. Note that $k \leq b$ and write $r = (b-k)p + (pk+j)$ and use the inductive assumption on $b-k$. Uniqueness is obvious. □

Representation dimension of Schur algebras. We claim the following:

4.11. **Proposition.** *For any u with $(p-1)|\delta| + 1 \leq u \leq (p-1)|\delta| + p$, there exists a partition $(p-1)\delta + w_0\lambda \in \Lambda(n,u)$ with $\lambda_1 < p$, such that*

$$\mathrm{End}_G(P((p-1)\delta + w_0\lambda)) \cong \Bbbk[x]/(x^n).$$

4.12. **Proposition.** *For any u with $0 \leq u \leq (p-1)|\delta|$ there exists a partition $\gamma \in \Lambda(n,u)$ with the additional property that γ is column p-regular and its first part is at most $(n-1)(p-1)$, such that*

$$\mathrm{End}_G(P(\gamma)) \cong \Bbbk[x]/(x^j)$$

for some j.

The proofs of these propositions will be given in subsequent sections.
Then we obtain the following theorem.

4.13. **Theorem.** *Expand r as in Lemma 4.10. Then there exist partitions $\lambda^i \in \Lambda(n, u_i - (p-1)\frac{n(n-1)}{2})$ $(0 \leq i \leq m-1)$ and $\gamma \in \Lambda(n, u_m)$ such that*

$$P((p^m-1)\delta + \sum_{0\leq i\leq m-1} p^i w_0(\lambda^i) + p^m\gamma)$$

is a projective injective $S(n,r)$-module with endomorphism ring

$$\Bbbk[x_0,\dots,x_m]/(x_i^{j_i} | 0 \leq i \leq m),$$

where $j_i = n$ for $i = 0,\dots m-1$, $j_m \geq 2$ if $u_m \geq p$ and $j_m = 1$ if $u_m \leq p-1$.

Proof. For every u_i $(0 \leq i \leq m-1)$, find a partition λ^i such that $(p-1)\delta + w_0\lambda^i \in \Lambda(n,u)$ is as in Proposition 4.11. For u_m, find a partition γ as in Proposition 4.12. Then the $(p-1)\delta + w_0\lambda^i$ and γ satisfy the assumptions of Lemma 4.8 and we obtain that

$$P((p^m-1)\delta + \sum_{0\leq i\leq m-1} p^i w_0\lambda^i + p^m\gamma) \cong (\bigotimes_{i=0}^{m-1} P((p-1)\delta + w_0\lambda^i)^{F^i}) \otimes P(\gamma)^{F^m}$$

is isomorphic to a tilting module and therefore projective injective. By Lemma 4.6

$$\begin{aligned}
&\mathrm{End}_G(P((p^m-1)\delta + \sum_{0\le i\le m-1} p^i w_0(\lambda^i) + p^m\gamma) \\
&\cong \mathrm{End}_G((\bigotimes_{i=0}^{m-1} P((p-1)\delta + w_0\lambda^i)^{F^i}) \otimes P(\gamma)^{F^m}) \\
&\cong (\bigotimes_{i=0}^{m-1} \mathrm{End}_G(P((p-1)\delta + w_0\lambda^i)) \otimes \mathrm{End}_G(P(\gamma))
\end{aligned}$$

where the last isomorphism comes from an iterated application of Lemma 4.6. Lemmas 4.11 and 4.12 now imply the result. □

Corollary 3.3 immediately implies

4.14. **Corollary.** *For*

$$p^{m+1} + \frac{p^m-1}{p-1} + (p^m-1)\frac{n(n-1)}{2} \le r \le p^{m+2} + \frac{p^{m+1}-1}{p-1} + (p^{m+1}-1)\frac{n(n-1)}{2} - 1,$$

$$\operatorname{repdim} S(n,r) \ge m+2.$$

5. Proofs of Proposition 4.11 and Proposition 4.12

Proof of Proposition 4.11. We first want to prove Proposition 4.11, i.e. construct a suitable partition μ for any given u with $(p-1)|\delta|+1 \le u \le (p-1)|\delta|+p$ such that $\mu = (p-1)\delta + w_0\lambda \in \Lambda(n,u)$ with $\lambda_1 < p$ and $\mathrm{End}_G(P(\mu) \cong \Bbbk[x]/(x^n).)$. Fix u. Set $b := u-(p-1)|\delta|$ with $1 \le b \le p$ and write $b = an+k$ with $1 \le k \le n < p$.

Consider the partition $\mu := (p-1)\delta + a\omega + k\epsilon^n$ and note that μ satisfies the requirements on the shape of the partition in Proposition 4.11 and that $d(\mu) = 0$ (as defined before Lemma 4.3). Note that by [9, Section 4, Theorem], the block of $S(n,u)$ containing μ is Morita equivalent to the block of $S(n,u-an)$ containing $\mu^1 := (p-1)\delta + k\epsilon^n$. In this case, Lemma 4.3 reduces to saying that the partitions in the same block as μ^1 are exactly those partitions with at most n parts which have the same p-core as μ. Considering the shape of this partition on an abacus, it is easy to see that the only partitions with at most n parts which have the same p-core are of the form $\mu^i := (p-1)\delta + k\epsilon^{n+1-i}$ for $1 \le i \le n$.

By Lemma 4.7 (b), $P(\mu^1) \cong T(\mu^n)$ and we now construct this module explicitly.

5.1. **Lemma.** *$\Delta(\mu^n)$ is uniserial with composition factors $L(\mu^n), L(\mu^{n-1}), \dots, L(\mu^1)$ read top to bottom.*

Proof. We proceed by induction on n. If $n=2$, it is well-known that $\Delta(p-1+k,0)$ has length 2 and composition factors $L(p-1+k,0)$ and $L(p-1,k)$.So let $n \ge 3$ and assume the statement holds for $n-1$, i.e. $\Delta_{n-1}((p-1)\delta^{n-1} + k\epsilon^1))$ is uniserial with composition factors by $L((p-1)\delta^{n-1}+k\epsilon^1), L((p-1)\delta^{n-1}+k\epsilon^2), \dots, L((p-1)\delta^{n-1}+k\epsilon^{n-1})$. Applying the $(p-1)$st power of the functor $\mathcal{J}_{n-1}$ and noting that $(p-1)\delta^{n-1} + (p-1)\omega^{n-1} = (p-1)\delta^n$, we obtain that

$$\mathcal{J}_{n-1}^{p-1}\Delta_{n-1}((p-1)\delta^{n-1} + \epsilon^1) = \Delta_{n-1}((p-1)\delta^n + k\epsilon^1)$$

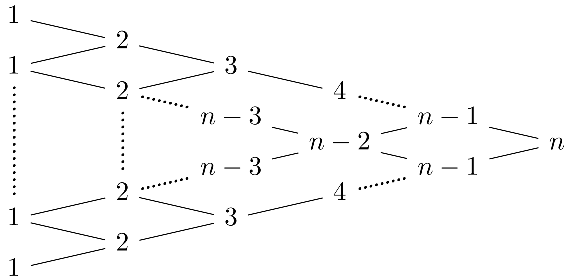
has a uniserial filtration with successive subquotients $L((p-1)\delta^n + k\epsilon^1), L((p-1)\delta^n + k\epsilon^2), \dots, L((p-1)\delta^n + k\epsilon^{n-1})$. On the other hand

$$\begin{aligned}
\Delta_{n-1}((p-1)\delta^n + k\epsilon^1) &\cong \mathcal{R}\Delta_n((p-1)\delta^n + k\epsilon^1) \\
&\cong e_{n-1}\Delta_n((p-1)\delta^n + k\epsilon^1) \\
&= e_{n-1}\Delta_n(\mu^n).
\end{aligned}$$

Hence $e_{n-1}\Delta(\mu^n)$ has a uniserial filtration by $L(\mu^n), L(\mu^{n-1}), \dots, L(\mu^2)$.

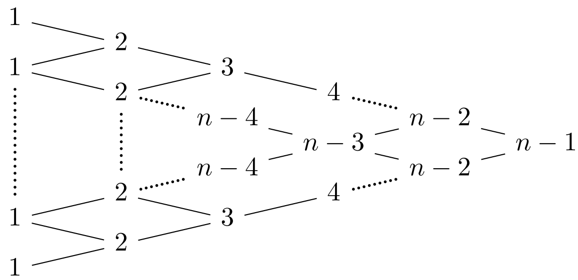
Now note that since $P(\mu^1) \cong T(\mu^n)$, it is self-dual and $\Delta(\mu^n)$ is a submodule of $T(\mu^n)$. Therefore, $\Delta(\mu^n)$ has a simple socle $L(\mu^1)$. Also, this must be the only occurrence of $L(\mu^1)$ as a composition factor of $\Delta(\mu^n)$, since $\Delta(\mu^n)$ appears only once in a Δ-filtration of $T(\mu^n)$ and hence $1 = [P(\mu^1) : \Delta(\mu^n)] = [\Delta(\mu^n) : L(\mu^1)]$. Hence all other composition factors of $\Delta(\mu^n)$ are of the form $L(\mu^i)$ for $i \geq 2$ and in particular correspond to partitions with at most $n-1$ parts. Since e_{n-1} acts as the identity on simple modules with at most $n-1$ parts and kills those with n parts, $e_{n-1}\Delta(\mu^n) \cong \Delta(\mu^n)/L(\mu^1)$. Knowing the uniserial filtration of $e_{n-1}\Delta(\mu^n)$ and extending with $L(\mu^1)$ in the unique way which produces a simple socle, we obtain the claim. □

5.2. **Lemma.** *$P(\mu^1)$ has composition structure*



In this picture, the number i in the picture stands for a composition factor $L(\mu^i)$, the horizontal layers correspond to the radical filtration and lines correspond to non-split extensions between composition factors.

Proof. Again, we prove this by induction on n. The case $n = 2$ is again well-known.So let $n \geq 3$ and assume the statement holds for $n-1$, i.e. $P_{n-1}((p-1)\delta^{n-1} + k\epsilon^{n-1})$ has composition structure



Observe that all composition factors $L(\nu)$ in this composition structure satisfy $\nu_1 < (p-1)(n-1)$, hence $P_{n-1}((p-1)\delta^{n-1} + k\epsilon^{n-1})$ factors over the quotient $S(n-1, (p-1)|\delta^{n-1}| + k)_{\leq_1(n-1)(p-1)}$. Using the equivalence from Fact (2), and noting that adding a first row of length $(p-1)(n-1)$ to $(p-1)\delta^{n-1} + k\epsilon^{n-1}$ gives μ^1, we obtain that the projective with label μ^1 for the algebra $S(n, |\mu^1|)_{=_1(p-1)(n-1)}$ has the same composition structure only now i stands for $L((p-1)\delta^n + k\epsilon^{n+1-i}) = L(\mu^i)$. On the other hand, the only partition in the block of $P(\mu^1)$ with first part strictly greater than $(n-1)(p-1)$ if μ^n, so $L(\mu^n)$ is the only composition factor occurring in $P_n(\mu^1)$ with first part greater than $(n-1)(p-1)$. It appears with multiplicity

$$\begin{aligned}
[P(\mu^1) : L(\mu^n)] &= \sum_{1 \leq i \leq n} [P(\mu^1) : \Delta(\mu^i)][\Delta(\mu^i) : L(\mu^n)] \\
&= [P(\mu^1) : \Delta(\mu^n)][\Delta(\mu^n) : L(\mu^n)] \\
&= [\Delta(\mu^n) : L(\mu^1)][\Delta(\mu^n) : L(\mu^n)] = 1
\end{aligned}$$

and generates the submodule $\Delta(\mu^n)$. Hence the projective with label μ^1 for the algebra $S(n, |\mu^1|)_{=1(p-1)(n-1)}$ is isomorphic to $P(\mu^1)/\Delta(\mu^n)$. Extending this module, whose composition structure we have determined above, by $\Delta(\mu^n)$ to obtain a self-dual module, can be only done in the way stated in the lemma. $\square$

5.3. **Corollary.** *For $0 \leq a \leq p-2$ and $1 \leq k \leq n$,*

$$\begin{aligned} &\mathrm{End}(P((p-1)\delta + a\omega + k\epsilon^n)) \\ &\qquad \cong \mathrm{End}(T((p-1)\delta + a\omega + k\epsilon^1)) \cong \Bbbk[x]/(x^n). \end{aligned}$$

Proof. This follows immediately from the lemma. $\square$

Setting $\lambda = a\omega + k\epsilon^1$, it is clear that $(p-1)\delta + w_0\lambda$ satisfies all requirements of Proposition 4.11 and this is hence proved.

Proof of Proposition 4.12. We now take care of Proposition 4.12, i.e. construct a partition $\gamma \in \Lambda(n, u)$, with $(p-1)|\delta|+1 \leq u \leq (p-1)|\delta|+p$, such that $\mathrm{End}_G(P(\gamma)) \cong \Bbbk[x]/(x^j)$ for some j, with the additional property that γ is column p-regular and its first part is at most $(n-1)(p-1)$. Note that if $u \leq p-1$, the module $P(\gamma)$ is simple for every $\gamma \in \Lambda(n, u)$, hence projective injective with endomorphism ring $\Bbbk$.

Let $d \leq n-1$ be the smallest number such that $(p-1)|\delta^d| < u \leq (p-1)|\delta^{d+1}|$ where δ^d is the partition $(d-1, d-2, \cdots, 1)$. Then write $x := u - (p-1)|\delta^d|$ in the form $x = ad - k$ with $a \geq 0$ and $1 \leq k \leq d \leq p-1$. Set $\gamma := (p-1)\delta^d + a\omega^d - k\epsilon^1$. Note that $0 \leq x \leq (p-1)d$, hence $a \leq p-1$ and γ is p-column regular. Note also that $1 \leq k \leq d \leq p-1$ implies that γ is a partition. By Lemma 4.4, we know that $P_s(\gamma)$ is tilting for $s \geq d+1$ and isomorphic to $T(\mathrm{Mull}(\gamma'))$. We now compute $\mathrm{Mull}(\gamma')$.

5.4. **Lemma.**

$$\mathrm{Mull}(\gamma') = \begin{cases} (p-1)\delta^d + a\omega^d - (0, \ldots, \underset{d}{k}) & \text{if } a \geq k \\ (p-1)\delta^d + a\omega^d - (0, \ldots, \underset{d-1}{k}, 0) & \text{if } a < k \end{cases}$$

Proof. Since μ^1 is column p-regular and $\mu_1^1 < (p-1)(n-1)$, we only need to check that $\mu^{n-1} = \mathrm{Mull}((\mu^1)')$. This can be done using the action of the crystal operators $\tilde{f}_i$ for the Lie algebra $\widehat{\mathfrak{sl}}_p$ on the set of p-regular partitions (see e.g. [13, Section 3]). To simplify notation and as we are not interested in the full fock space we write f_i for the crystal operator $\tilde{f}_i$. It is then easy to see that $(p-1)(\delta^d)'$ can be written as

$$(p-1)(\delta^d)' = (f_d^{d-1} \cdots f_{d-2}^{d-1}) \cdots (f_3^2 \cdots f_{p-1}^2 f_0^2 f_1^2)(f_2 \cdots f_{p-1} f_0)\emptyset.$$

Similarly $((p-1)\delta^d + a\omega^d)'$ can be written as

$$((p-1)\delta^d + a\omega^d)' = (f_{d-a}^d \cdots f_{d-1}^d)(f_d^{d-1} \cdots f_{d-2}^{d-1}) \cdots (f_3^2 \cdots f_{p-1}^2 f_0^2 f_1^2)(f_2 \cdots f_{p-1} f_0)\emptyset.$$

Since in γ' we have taken away k boxes in the first column of $((p-1)\delta^d - a\omega^{d-1})'$, which is the last to be filled up, γ' has the same expression as $(p-1)(\delta^d)' - a\omega^{d-1}$ except that the exponent of th last k crystal operators is smaller by one, i.e.

$$\gamma' = \begin{cases} (f_{d-a}^{d-1} \cdots f_{d-a+k-1}^{d-1} f_{d-a+k}^d \cdots f_{d-1}^d)(f_d^{d-1} \cdots f_{d-2}^{d-1}) \cdots & \\ \qquad \cdots (f_3^2 \cdots f_{p-1}^2 f_0^2 f_1^2)(f_2 \cdots f_{p-1} f_0)\emptyset & \text{if } a \geq k \\ (f_{d-a}^{d-1} \cdots f_{d-1}^{d-1})(f_d^{d-2} \cdots f_{d+k-a-1}^{d-2} f_{d+k-a}^{d-1} \cdots f_{d-2}^{d-1}) \cdots & \\ \qquad \cdots (f_3^2 \cdots f_{p-1}^2 f_0^2 f_1^2)(f_2 \cdots f_{p-1} f_0)\emptyset & \text{if } a < k. \end{cases}$$

By [12]

$$\mathrm{Mull}(\gamma') = \begin{cases} (f^{d-1}_{p-(d-a)} \cdots f^{d-1}_{p-(d-a+k-1)} f^{d}_{p-(d-a+k)} \cdots f^{d}_{p-d+1})(f^{d-1}_{p-d} \cdots f^{d-1}_{p-(d-2)}) \cdots \\ \qquad \cdots (f^2_{p-3} \cdots f^2_1 f^2_0 f^2_{p-1})(f_{p-2} \cdots f_1 f_0)\emptyset \qquad \text{if } a \geq k \\ (f^{d-1}_{p-(d-a)} \cdots f^{d-1}_{p-(d-1)})(f^{d-2}_{p-d} \cdots f^{d-2}_{p-(d+k-a-1)} f^{d-1}_{p-(d+k-a)} \cdots f^{d-1}_{p-(d-2)}) \cdots \\ \qquad \cdots (f^2_{p-3} \cdots f^2_1 f^2_0 f^2_{p-1})(f_{p-2} \cdots f_1 f_0)\emptyset \qquad \text{if } a < k. \end{cases}$$

Consider first the case $a \geq k$. Applying the first f_j to $\emptyset$ up to

$$f^{d}_{p-(d-a+k)} \cdots f^{d}_{p-d+1})(f^{d-1}_{p-d} \cdots f^{d-1}_{p-(d-2)}) \cdots (f^2_{p-3} \cdots f^2_1 f^2_0 f^2_{p-1})(f_{p-2} \cdots f_1 f_0)\emptyset$$

this forms $(p-1)\delta^d + (a-k)\omega^d$, then we add $d-1$ boxes of content $p-(d-a+k-1)$, however, there are d addable ones (one in each row). Higher rows have priority in the algorithm, hence the last row does not get a new box in this step. Now for the next $k-1$ indices, there are only $d-1$ addable boxes of this content (one in each of the first $d-1$ rows, but none in the last row, as it ends with a box of content $p-(d-a+k)$), hence we add them all, and obtain the partition $(p-1)\delta^d + a\omega^d - k\epsilon^d$ as claimed. In case $a < k$, up to $f^{d-1}_{p-(d+k-a)} \cdots f^{d-1}_{p-(d-2)}) \cdots (f^2_{p-3} \cdots f^2_1 f^2_0 f^2_{p-1})(f_{p-2} \cdots f_1 f_0)\emptyset$ we have built the partition $\delta^{d-2} + ((p-1)-(k-a))\omega^{d-1}$. This has $d-1$ addable boxes of content $p-(d+k-a-1)$, but we only add $d-2$, hence none in row $d-1$. We then add boxes of contents $p-(d+k-a)+2$ up to $p-d$, $d-1$ of each, which is also the maximal number since in each step there is an addable box in rows 1 up to $d-2$ but not in row $d-1$. After this, we have $d-1$ addable $p-(d-1)$ boxes, one in each of the rows 1 up to $d-2$ and also the first box in row d. Then we continue to add boxes to all of these rows (1 up to $d-2$ and d) in every step, until we're done. We cannot add any more boxes to row $d-1$, since the next box would have to have content $p-(d+k-a-1)$, which does not occur any more. Hence $\mathrm{Mull}(\gamma') = (p-1)\delta^d + a\omega^d - k\epsilon^{d-1}$ in this case as stated. □

Since $P_{d+1}(\gamma) \cong T_{d+1}(\mathrm{Mull}(\gamma'))$, we know that all partitions ν, such that $\Delta(\nu)$ appears in a Δ-filtration of $P_{d+1}(\gamma)$, must satisfy $\gamma \leq \nu \leq \mathrm{Mull}(\gamma')$. Again, Lemma 4.3 yields that these partitions ν must have the same p-core as γ. Considering γ in abacus notation, it is again easy to see that they ν must be of the form $\nu_i(d) = (p-1)\delta^d + a\omega^d - k\epsilon^i$ for $1 \leq i \leq d$ if $a \geq k$ and $1 \leq i \leq d-1$ if $a < k$. Let $s(d) = d$ if $a \geq k$ and $s(d) = d-1$ if $a < k$. Since none of these partitions have more than d rows, $P_d(\gamma)$ has the same submodule lattice and is also projective injective and isomorphic to $T_d(\mathrm{Mull}(\gamma'))$. We now show that $P_d(\gamma)$ is of a similar shape as $P(\mu^1)$ earlier.

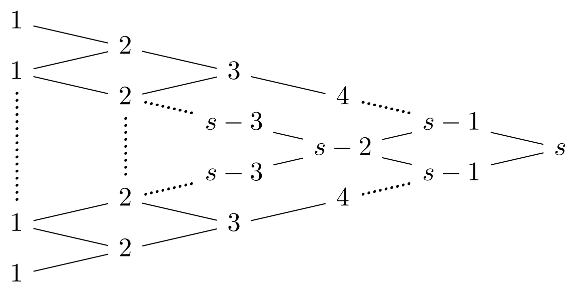
5.5. **Lemma.** $\Delta_d(\mathrm{Mull}(\gamma'))$ *is uniserial with successive subquotients* $L(\mathrm{Mull}(\gamma')) = L(\nu_{s(d)}(d)), L(\nu_{s(d)-1}(d)), \ldots, L(\nu_1(d)) = L(\gamma)$.

Proof. To see this, we again use induction on d. The case $d = 2$ is trivial since, in case $a \geq k$, $\Delta_2(p-1+a, a-k)$ has composition factors $L(p-1+a, a-k)$ and $L(p-1+a-k, a)$. In case $a < k$, $s(2) = 1$ and $\Delta_2(\nu_{s(1)}) = \Delta_2(\nu_1) = L(\nu_{s(1)})$ is simple.

Assume the statement is true for $d-1$, i.e. $\Delta_{d-1}(\nu_{s(d-1)}(d-1))$ is uniserial with composition factors $L(\nu_{s(d-1)}(d-1)), L(\nu_{s(d-1)-1}(d-1)), \ldots, L(\nu_1(d-1))$. All of these partitions $\nu_i(d-1) = \delta^{d-1} + a\omega^{d-1} - k\epsilon^i$ have first row of length less than or equal to $(p-1)(d-1)+a$, thus the action of $S(d-1, |\nu_i(d-1)|)$ on $\Delta_{d-1}(\nu_{s(d-1)}(d-1))$ factors over $S(d-1, |\nu_i(d-1)|)_{\geq 1 (p-1)(d-1)+a}$ which is isomorphic to $S(d, |\nu_i(d)|)_{=1 (p-1)(d-1)+a}$ (observe $|\nu_i(d-1)| + (p-1)(d-1) + a = |\nu_i(d)|$). The equivalence from Fact (2) implies that the standard module corresponding to $\nu_{s(d)}(d)$ for this latter algebra has a uniserial filtration with successive composition factors $L(\nu_{s(d)}(d))$, $L(\nu_{s(d)-1}(d))$, …, $L(\nu_2(d))$.

On the other hand, by the same argument as in Lemma 5.1, $\Delta_d(\nu_{s(d)})$ has simple socle $L(\nu_1(d))$ and $\Delta_d(\nu_{s(d)}(d))/L(\nu_1(d))$ has no further composition factors $L(\nu_1(d))$, and all other composition factors correspond to a partition with first row of length $(p-1)(d-1)+a$. Therefore, $\Delta_d(\nu_{s(d)}(d))/L(\nu_1(d))$ is isomorphic to the standard module corresponding to $\nu_{s(d)}(d)$ for $S(d,|\nu_i(d)|)_{=_1(p-1)(d-1)+a}$ which, as seen above, has a uniserial filtration. Extending by $L(\nu_1(d))$ to a module with simple socle, we obtain exactly the desired uniserial filtration of $\Delta_d(\nu_s(d))$. □

5.6. **Lemma.** *Set $s = s(d)$. Then $P_d(\gamma)$ has composition structure of the form*



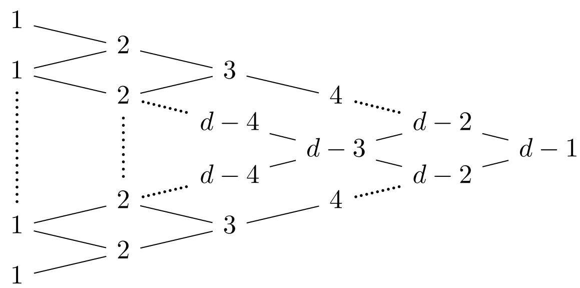
where the picture is to be read as in Lemma 5.2, except that i now stands for $L(\nu_i(d))$

Proof. Again the statement is well-known for $d = 2$, as described in the previous Lemma.

So let $d \geq 3$ and assume it is true for $d-1$. Since

$$(p-1)\delta^d - (k) = (p-1)\delta^{d-1} + a\omega^{d-1} - k\epsilon^1$$

is of the form $\nu_1(d-1)$ for $a = p-1$, we can assume that $P_{d-1}((p-1)\delta^d - k\epsilon^1)$, as well as $P_d((p-1)\delta^d - k\epsilon^1)$ by the argument preceding Lemma 5.5, has composition structure



where i stands for a composition factor $L((p-1)\delta^d - k\epsilon^i)$.

Case 1: $a \geq k$

By the isomorphism $S(d,|(p-1)\delta^d - (k)|) \cong S(d,|\gamma|)_{\leq_{d-1}|\gamma|-a}$ from a-fold application of Fact (3) (observe $|(p-1)\delta^d - (k)| = \gamma - ad$), the projective module corresponding to γ for this latter algebra has the same composition structure only now i stands for $L((p-1)\delta^d + a\omega^d - k\epsilon^i) = L(\nu_i(d))$. On the other hand note that the only composition factor $L(\lambda)$ of $P_d(\gamma)$ which does not satisfy the requirement $\lambda_n \geq a$ is $L(\nu_d)$ which generates the submodule $\Delta(\nu_d)$. Hence $P_d(\gamma)/\Delta(\nu_d)$ is the projective module indexed by γ for the algebra $S(d,|\gamma|)_{\leq_{d-1}|\gamma|-a}$. Again extending with $\Delta(\nu_d)$ to a self-dual module can only be done in the way claimed in the Lemma.

Case 2: $a < k$

In this case, every ν_i has dth row of length a, hence $P_d(\gamma)$ is a module for $S(d,|\gamma|)_{\leq_{d-1}|\gamma|-a}$, and has the same submodule lattice as the projective with label

γ for this algebra. Again by the same isomorphism

$$S(d, |\gamma|)_{\leq_{d-1}|\gamma|-a} \cong S(d, |(p-1)\delta^d| - k),$$

we obtain that $P_d(\gamma)$ has the same composition structure as $P_d((p-1)\delta^d - k\epsilon^1)$, which is given above, only that in the composition structure of $P_d(\gamma)$, i stands for $L((p-1)\delta^d + a\omega - k\epsilon^i) = L(\nu_i(d))$, hence we're done. □

5.7. **Corollary.** $\mathrm{End}_G(P_n(\gamma)) \cong \begin{cases} \Bbbk[x]/(x^d) & \textit{if } a \geq k \\ \Bbbk[x]/(x^{d-1}) & \textit{if } a < k \end{cases}$.

Proof. This follows immediately from Lemma 5.6 and the fact that $\mathrm{End}_G(P_n(\gamma)) \cong \mathrm{End}_G(P_d(\gamma))$ by $S(d,r) \cong f_{d,r} S(n,r) f_{d,r}$. □

Therefore γ as constructed satisfies the requirements of Proposition 4.12 and we have completed the prove of the latter.

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