WILD ALGEBRAS HAVE ONE-POINT EXTENSIONS OF REPRESENTATION DIMENSION AT LEAST FOUR

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Abstract. We show that any wild algebra has a one-point extension of representation dimension at least four, and more generally that it has an n-point extension of representation dimension at least $n+3$. We give two explicit constructions, and obtain new examples of small algebras of representation dimension four.

1. Introduction

In his Queen Mary College Notes [1], Auslander defined the representation dimension of an artin algebra as follows.

1.1. **Definition** (Auslander).

$$\operatorname{repdim} \Lambda = \min\{\operatorname{gld} \operatorname{End}_\Lambda(M) \mid M \in \Lambda\text{-mod is a generator and a cogenerator}\}.$$

The motivation for this definition is the following result, which he proved in the same paper.

1.2. **Theorem** (Auslander).

$$\Lambda \text{ } \textit{representation finite} \iff \operatorname{repdim} \Lambda \leq 2.$$

Auslander hoped that the value of the representation dimension of a representation infinite algebra is a good measure of how far this algebra is from having finite representation type (see [1], Chapter III, Section 5).

The distinction between tame and wild representation type is another way of saying "how infinite" the representation theory of an algebra is. It is therefore natural to ask for connections between these two. It has been conjectured or asked by many people studying this subject (including Holm, Iyama, Reiten, Schröer), whether the following implication holds (see [3] for a partial result).

1.3. **Conjecture.**

$$\Lambda \text{ } \textit{tame} \overset{?}{\Rightarrow} \operatorname{repdim} \Lambda \leq 3.$$

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Note that in general the converse does not hold,

$$\operatorname{repdim} \Lambda \leq 3 \not\Rightarrow \Lambda \text{ tame},$$

any wild hereditary algebra being a counterexample. Here we want to prove the following (necessarily weaker) result. For the notation see Section 2.

1.4. **Theorem.** *Let Λ be an algebra of wild representation type. Then there is a one-point extension $\Lambda[M]$ such that* $\operatorname{repdim} \Lambda[M] \geq 4$.

We will give two methods for constructing such one-point extensions of representation dimension at least four.

The first one will be explained in Section 4 and proven to work in Sections 5 to 8. For a wild algebra Λ there is, by definition, a two parameter family of indecomposable modules. We may consider this family as one $\Lambda \otimes_k k[X, Y]$-module L. We give a criterion (Theorem 4.2) when a finite dimensional Λ-submodule of L gives rise to a one-point extension of representation dimension at least four. Then we show that, provided the base field is large enough, for any suitable chain of finite dimensional submodules of L this will eventually hold (Theorem 4.3). In Section 9 we apply this method to some small wild algebras. By doing so we will obtain new examples of algebras of representation dimension four (see Table 1).

The second method is presented in Section 10. We choose some algebra Λ_0, which is known to have a one-point extension of representation dimension four (and has certain additional properties). Then for any given wild algebra Λ we use a representation embedding from Λ_0-modules to Λ-modules to create a one-point extension of Λ, which has representation dimension at least four. This not only works for one-point extensions and representation dimension four, but we obtain the following more general result, which allows the construction of new examples of algebras of arbitrarily large representation dimension.

1.5. **Theorem.** *Let Λ be of wild representation type, $n \in \mathbb{N}$. Then there is an n-point extension $\Lambda[M_1][M_2] \cdots [M_n]$ of representation dimension at least $n + 3$.*

2. Notation

We always assume k to be a field.

For a k-algebra Λ we denote the category of finitely generated left Λ-modules by Λ-mod, and the category of Λ-modules of finite k-dimension by Λ-fd. We will mostly assume Λ to be a finite dimensional algebra, in which case these two notions coincide.

2.1. **Definition** (one-point extension)**.** Let Λ be a k-algebra. For $M \in \Lambda$-fd we will denote the *one-point extension* $\left(\begin{smallmatrix} k & 0 \\ M & \Lambda \end{smallmatrix}\right)$ by $\Lambda[M]$.

The $\Lambda[M]$ modules are of the form $\binom{X_0}{X_1}_\varphi$ with $X_0 \in k\text{-Mod}$, $X_1 \in \Lambda\text{-Mod}$ and $\varphi : M \otimes_k X_0 \longrightarrow X_1$. We will usually omit the φ when there is no chance of confusion.

3. Lattices, representation embeddings and representation dimension

3.1. **Definition** (right lattice)**.** Let Λ and R be k-algebras. A $\Lambda \otimes_k R^{\text{op}}$-module L (that is a Λ-R-bimodule, on which k acts centrally) is called *right lattice* if it is finitely generated projective as right R-module. The category of right $\Lambda \otimes_k R^{\text{op}}$-lattices will be denoted by $\Lambda \otimes_k R^{\text{op}}\text{-r.lat}$. When no sides are mentioned, all lattices will be assumed to be right lattices, that is finitely generated projective with respect to the ring acting from the right (this ring will usually be denoted by R).

Note that any $L \in \Lambda \otimes_k R^{op}\text{-r.lat}$ induces an exact functor

$$L \otimes_R - : R\text{-fd} \longrightarrow \Lambda\text{-fd}\,.$$

3.2. **Definition** (representation embedding)**.** For $L \in \Lambda \otimes_k R^{\text{op}}\text{-r.lat}$ we say that L *induces a representation embedding* if the functor $L \otimes_R -$ preserves indecomposability and reflects isomorphism classes.

3.3. **Definition** (wild)**.** A k-algebra Λ is called *wild* if it satisfies the following equivalent conditions (see for instance [6, pages 37-40]).

(1) For any finitely generated k-algebra R there is a representation embedding $R\text{-fd} \xrightarrow{L\otimes_R -} \Lambda\text{-fd}$.
(2) For any finite dimensional k-algebra R there is a representation embedding $R\text{-mod} \xrightarrow{L\otimes_R -} \Lambda\text{-fd}$.
(3) For $R = k[X, Y]$, the polynomial ring in two variables, there is a representation embedding $R\text{-fd} \xrightarrow{L\otimes_R -} \Lambda\text{-fd}$.

The third equivalent condition in the definition above allows us to mostly assume $R = k[X, Y]$. We denote by $\overline{k}$ the algebraic closure of k. To (hopefully) simplify notation, for $\alpha, \beta \in \overline{k}$ we will denote the extension field $k[\alpha, \beta]$ by $k_{\alpha\beta}$. Moreover for any k-vector space M we will denote by $M_{\alpha\beta}$ the $k_{\alpha\beta}$-vector space $M \otimes_k k_{\alpha\beta}$. Note that $M_{\alpha\beta}$ inherits all additional structure from M; for instance if Λ is a k algebra then $\Lambda_{\alpha\beta}$ is a $k_{\alpha\beta}$-algebra.

3.4. **Definition** (full rank sublattice)**.** Let Λ be a k-algebra. For a Λ-submodule L' of a $\Lambda \otimes_k k[X, Y]$-lattice L we say that L' *generates a full rank sublattice*, if one of the following equivalent conditions is satisfied:

(1) $\text{rk}_{k[X,Y]}(L' \cdot k[X, Y]) = \text{rk}_{k[X,Y]} L$,
(2) L' contains $\text{rk}_{k[X,Y]} L$ elements which are $k[X, Y]$-linearly independent,
(3) the multiplication map $L' \otimes_k k(X, Y) \longrightarrow L \otimes_{k[X,Y]} k(X, Y)$ is onto,

(4) the set of all $(\alpha, \beta) \in \overline{k}^2$, such that the composition

$$L'_{\alpha\beta} \hookrightarrow L_{\alpha\beta} \twoheadrightarrow L_{\alpha\beta}/(X-\alpha, Y-\beta)$$

is onto, is non-empty and Zariski open.

(To see that Condition (4) is equivalent to the others, note that $\dim_{k_{\alpha\beta}} L_{\alpha\beta}/(X-\alpha, Y-\beta) = \operatorname{rk} L$, and moreover that a finite $k[X,Y]$-linear independent set in L is mapped to a linear independent set in $L_{\alpha\beta}/(X-\alpha, Y-\beta)$ for all (α, β) in a non-empty open subset of $\overline{k}^2$.)

Our tool for establishing lower bounds for the representation dimension is the following.

3.5. **Theorem** (a special case of [5, 4.9])**.** *Let Λ be a finite dimensional algebra and $d \in \mathbb{N}$. Set $R = k[X_1, \ldots, X_d]$ and let L be a $\Lambda \otimes_k R$-lattice. Assume the set*

$$\{\mathfrak{p} \in \operatorname{MaxSpec} R \mid (L \otimes_R -)_{\operatorname{Ext}^d} \operatorname{Ext}^d_R(R/\mathfrak{p}, R/\mathfrak{p}) \neq 0\}$$

is Zariski-dense. Then

$$\operatorname{repdim} \Lambda \geq d + 2.$$

(Here, as in [5], the index Ext^d is supposed to emphasize that $L \otimes_R -$ turns d-extensions into d-extensions, and we do not apply $L \otimes_R -$ to the R-module $\operatorname{Ext}^d_R(R/\mathfrak{p}, R/\mathfrak{p})$.)

4. Construction of one-point extensions of representation dimension four

In this section we give our main method of constructing one-point extensions of wild algebras, which have representation dimension at least four. Theorem 4.2 gives a criterion for the representation dimension of certain one-point extensions to be at least four, and Theorem 4.3 ensures that, provided the base field is large enough, we will always be able to satisfy the assumptions of this criterion.

Throughout this section Λ is assumed to be a finite dimensional algebra.

4.1. **Setup.** Let L be a $\Lambda \otimes_k k[X,Y]$-lattice. We will mostly think of L inducing a representation embedding, but it is only necessary to assume this in Theorem 4.3.

We choose a Λ-submodule L' of L, which is finite dimensional (but otherwise arbitrary for the moment).

For $(\alpha, \beta) \in \overline{k}^2$ let $f_{\alpha\beta}$ be the composition

$$L'_{\alpha\beta} \cap L_{\alpha\beta}(X-\alpha, Y-\beta) \hookrightarrow L_{\alpha\beta}(X-\alpha, Y-\beta) \longrightarrow L_{\alpha\beta}/(X-\alpha, Y-\beta),$$

where the right factor is induced by the map of $k[X,Y]$-modules

$$\begin{aligned} k_{\alpha\beta}[X,Y](X-\alpha,Y-\beta) &\twoheadrightarrow k_{\alpha\beta}[X,Y]/(X-\alpha,Y-\beta) \\ (X-\alpha)^i(Y-\beta)^j &\longmapsto \begin{cases} 1 & (i,j)=(0,1) \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Moreover, let $\pi_{\alpha\beta}$ be the map

$$L_{\alpha\beta}/((X-\alpha)^2,Y-\beta) \twoheadrightarrow L_{\alpha\beta}/(X-\alpha,Y-\beta)$$

induced by

$$k_{\alpha\beta}[X,Y]/((X-\alpha)^2,Y-\beta) \twoheadrightarrow k_{\alpha\beta}[X,Y]/(X-\alpha,Y-\beta).$$

4.2. **Theorem.** *Let Λ be a finite dimensional k-algebra, and L a $\Lambda\otimes_k k[X,Y]$-lattice. Let L' be a finite dimensional Λ-submodule of L which generates a full rank sublattice, such that the set*

$$\{(\alpha,\beta)\in\overline{k}^2 \mid f_{\alpha\beta} \text{ does not factor through } \pi_{\alpha\beta}\}$$

is Zariski-dense. Then

$$\operatorname{repdim}\Lambda[L'] \geq 4.$$

The next theorem makes sure that, under suitable conditions, there are Λ-submodules L' satisfying the assumption of Theorem 4.2.

4.3. **Theorem.** *Let Λ be a finite dimensional k-algebra, and assume k is not countable. Let L be a $\Lambda\otimes_k k[X,Y]$-lattice inducing a representation embedding. For any chain $(L^i)_{i\in\mathbb{N}}$ of finite dimensional Λ-submodules of L (that is $L^1\subseteq L^2\subseteq L^3\subseteq\cdots$) such that $L=\cup_i L^i$, there is $i\in\mathbb{N}$ such that for $L'=L^i$ the set*

$$\{(\alpha,\beta)\in k^2 \mid f_{\alpha\beta} \text{ factors through } \pi_{\alpha\beta}\}$$

is not Zariski-dense in k^2.

From Theorems 4.2 and 4.3 we immediately obtain

4.4. **Corollary.** *Let Λ be a finite dimensional algebra over an uncountable field k. Then Λ has a one-point extension of representation dimension at least four.*

Proof. Note that the complement of a non-dense subset of k^2 contains a non-empty open subset of k^2, and any non-empty subset of k^2 is dense in $\overline{k}^2$. $\square$

5. Proof of Theorem 4.2

We will apply Theorem 3.5. To do so we need a $\Lambda[L']\otimes_k k[X,Y]$-lattice. By assumption we have a $\Lambda\otimes_k k[X,Y]$-lattice L. From this we obtain the $\Lambda[L']\otimes_k k[X,Y]$-lattice

$$\widehat{L}=\begin{pmatrix} k[X,Y] \\ L \end{pmatrix}_{\varphi},$$

where the map $\varphi : L' \otimes_k k[X,Y] \longrightarrow L$ is just multiplication in L.

To verify the assumptions of Theorem 3.5, we set

$$\mathbb{E}_{\alpha\beta} : \frac{k_{\alpha\beta}[X,Y]}{(X-\alpha, Y-\beta)} \rightarrowtail \frac{k_{\alpha\beta}[X,Y]}{((X-\alpha)^2, Y-\beta)} \to \frac{k_{\alpha\beta}[X,Y]}{(X-\alpha, (Y-\beta)^2)} \twoheadrightarrow \frac{k_{\alpha\beta}[X,Y]}{(X-\alpha, Y-\beta)},$$

where the first two maps are induced by multiplication with $X-\alpha$ and $Y-\beta$, respectively. We investigate when $\widehat{L} \otimes_{k[X,Y]} \mathbb{E}_{\alpha\beta}$ splits as 2-extension of Λ-modules. This 2-extension is represented by

$$\begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \rightarrowtail \begin{pmatrix} k_{\alpha\beta}\oplus k_{\alpha\beta}(X-\alpha) \\ \frac{L_{\alpha\beta}}{((X-\alpha)^2,Y-\beta)} \end{pmatrix} \longrightarrow \begin{pmatrix} k_{\alpha\beta}\oplus k_{\alpha\beta}(Y-\beta) \\ \frac{L_{\alpha\beta}}{(X-\alpha,(Y-\beta)^2)} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix}.$$

We fix $(\alpha,\beta) \in \overline{k}^2$ such that the composition $L'_{\alpha\beta} \hookrightarrow L_{\alpha\beta} \twoheadrightarrow L_{\alpha\beta}/(X-\alpha, Y-\beta)$ is epi (note that there are such pairs (α,β) by Condition (4) in Definition 3.4). Then we turn the above 2-extension into a 1-extension by

$$\operatorname{Ext}^2\left(\begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix}, \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix}\right) = \operatorname{Ext}^1\left(\Omega\begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix}, \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix}\right),$$

where Ω denotes the syzygy as $\Lambda[L']_{\alpha\beta}$-module. In our situation, that means we have to find out whether the short exact sequence

$$\begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \rightarrowtail H \twoheadrightarrow \begin{pmatrix} 0 \\ L'_{\alpha\beta}\cap L_{\alpha\beta}(X-\alpha,Y-\beta) \end{pmatrix} \tag{1}$$

in the following diagram splits.

$$\begin{tikzcd}
\begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \arrow[r, tail] \arrow[dd, equal] & \begin{pmatrix} k_{\alpha\beta}\oplus k_{\alpha\beta}(X-\alpha) \\ \frac{L_{\alpha\beta}}{((X-\alpha)^2,Y-\beta)} \end{pmatrix} \arrow[rr] \arrow[dr, two heads] & & \begin{pmatrix} k_{\alpha\beta}\oplus k_{\alpha\beta}(Y-\beta) \\ \frac{L_{\alpha\beta}}{(X-\alpha,(Y-\beta)^2)} \end{pmatrix} \arrow[r, two heads] & \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \arrow[dd, equal] \\
& \arrow[dr, phantom, "\mathrm{PB}"] & \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \arrow[ur, tail] & & \\
\begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \arrow[r, tail] & H \arrow[uu, "\begin{pmatrix} ? \\ \pi_{\alpha\beta} \end{pmatrix}"'] \arrow[rr] \arrow[dr, two heads] & & \begin{pmatrix} k_{\alpha\beta} \\ L'_{\alpha\beta} \end{pmatrix} \arrow[uu] \arrow[r, two heads] & \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix} \\
& & \begin{pmatrix} 0 \\ L'_{\alpha\beta}\cap L_{\alpha\beta}(X-\alpha,Y-\beta) \end{pmatrix} \arrow[uu, crossing over, "\begin{pmatrix} 0 \\ f_{\alpha\beta} \end{pmatrix}"'] \arrow[ur, tail] & &
\end{tikzcd}$$

Here the first row is the original 2-extension, the short exact sequence

$$\begin{pmatrix} 0 \\ L'_{\alpha\beta}\cap L_{\alpha\beta}(X-\alpha,Y-\beta) \end{pmatrix} \rightarrowtail \begin{pmatrix} k_{\alpha\beta} \\ L'_{\alpha\beta} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} k_{\alpha\beta} \\ \frac{L_{\alpha\beta}}{(X-\alpha,Y-\beta)} \end{pmatrix}$$

is the projective resolution of $\begin{pmatrix} k_{\alpha\beta} \\ L_{\alpha\beta}/(x-\alpha,Y-\beta) \end{pmatrix}$ (note that $\begin{pmatrix} k_{\alpha\beta} \\ L'_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} k \\ L' \end{pmatrix}^{[k_{\alpha\beta}:k]}$ as Λ-modules), and H is the pullback of the square to its right. By assumption there is a Zariski-dense $\mathcal{U} \subseteq \overline{k}^2$, such that $f_{\alpha\beta}$ does not factor through $\pi_{\alpha\beta}$ for any $(\alpha,\beta) \in \mathcal{U}$. Therefore the short exact sequence

(1) is not split for any of these (α, β). This means that our original 2-extension $\widehat{L} \otimes_{k[X,Y]} \mathbb{E}_{\alpha\beta}$ is non-split for all

$$(\alpha, \beta) \in \mathcal{U} \setminus \{(\alpha, \beta) \mid L'_{\alpha\beta} \hookrightarrow L_{\alpha\beta} \twoheadrightarrow L_{\alpha\beta}/(X-\alpha, Y-\beta) \text{ is not epi}\}.$$

The set subtracted is a proper closed subset by the assumption that L' generates a full rank sublattice. Therefore the difference is still dense.

Hence the assumptions of Theorem 3.5 are met, so $\operatorname{repdim} \Lambda[L'] \geq 4$. □

6. Limits and completeness

In this section we recall some classical results in order to fix notation and for the convenience of the reader. They will be applied to representation embeddings in Section 7 and used in the proof of Theorem 4.3 in Section 8.

Throughout this section we assume k to be a field, and R to be a noetherian k-algebra.

6.1. **Lemma.** *Let M be a finite dimensional R-module,*

$$\cdots \longrightarrow N_3 \longrightarrow N_2 \longrightarrow N_1$$

a sequence of morphisms of finite dimensional R-modules. Then for any j we have

$$\operatorname{Ext}^j_R(M, \varprojlim_i N_i) = \varprojlim_i \operatorname{Ext}^j_R(M, N_i).$$

Proof. See [7], Section 3.5, in particular Proposition 3.5.7 and Theorem 3.5.8. □

6.2. **Lemma.** *Let M be a finite dimensional R-module,*

$$N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow \cdots$$

a sequence of morphisms of finite dimensional R-modules. Then for any j we have

$$\operatorname{Ext}^j_R(\varinjlim_i N_i, M) = \varprojlim_i \operatorname{Ext}^j_R(N_i, M).$$

Proof. This is just the opposite version of Lemma 6.1. □

6.3. **Definition.** We call an ideal $I \trianglelefteq R$ *cofinite* if $\dim_k R/I < \infty$. We call R *complete* with respect to an ideal I if $R = \varprojlim_n R/I^n$.

6.4. **Lemma.** *Let R be complete with respect to a cofinite ideal I, and M be a finitely generated R-module. Then $M = \varprojlim_n M/I^nM$.*

7. Representation embeddings and infinite dimensional modules

In this section we assume R and S to be noetherian k-algebras, and that there is a representation embedding $R\text{-fd} \xrightarrow{L\otimes_R -} S\text{-fd}$.

Our aim is to show that the functor $L \otimes_R -$ also reflects splitting of certain short exact sequences involving infinite dimensional modules. More precisely, we want to show the following:

7.1. **Theorem.** *Let R be complete with respect to some cofinite ideal or be commutative. Let $A \rightarrowtail B \twoheadrightarrow C$ be a non-split short exact sequence of R-modules, with $\dim_k A < \infty$ or $\dim_k C < \infty$. Then the induced exact sequence*

$$L \otimes_R A \rightarrowtail L \otimes_R B \twoheadrightarrow L \otimes_R C$$

is also non-split.

We will prove the claims in Lemmas 7.2 to 7.5.

7.2. **Lemma.** *Theorem 7.1 holds if R is complete with respect to some cofinite ideal I, A and B are finitely generated, and $\dim_k C < \infty$.*

Proof. By Lemma 6.4 we have $A = \varprojlim_n A/I^n A$. Therefore, by Lemma 6.1 we have

$$\operatorname{Ext}^1_R(C, A) = \operatorname{Ext}^1_R(C, \varprojlim_n A/I^n A) = \varprojlim_n \operatorname{Ext}^1_R(C, A/I^n A).$$

So there is $n \in \mathbb{N}$ such that the pushout of $A \rightarrowtail B \twoheadrightarrow C$ along $A \twoheadrightarrow A/I^n A$ is also non-split. This pushout however is a short exact sequence of finite dimensional R-modules. Therefore it remains non-split when tensored with L. Tensoring the entire pushout with L we obtain

$$\begin{tikzcd}
L \otimes_R A \arrow[r, tail] \arrow[d, two heads] & L \otimes_R B \arrow[r, two heads] \arrow[d] & L \otimes_R C \arrow[d, equal] \\
L \otimes_R A/I^n A \arrow[r, tail] & L \otimes_R \mathrm{PO} \arrow[r, two heads] & L \otimes_R C
\end{tikzcd}$$

Since the lower sequence is non-split, and it is the pushout of the upper sequence, the upper sequence also has to be non-split. $\square$

7.3. **Lemma.** *Theorem 7.1 holds if R commutative, A and B are finitely generated and $\dim_k C < \infty$.*

Proof. Let I be a cofinite ideal of R such that I^n annihilates C for some $n \in \mathbb{N}$. Set $\widehat{R} = \varprojlim_i R/I^i$ the completion of R at I. Then the map

$$\begin{aligned}
\widehat{R} \otimes_R \operatorname{Hom}_R(M, N) &\longrightarrow \operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R M, \widehat{R} \otimes_R N) \\
\hat{r} \otimes \varphi &\longmapsto [\hat{r}' \otimes m \mapsto \hat{r}\hat{r}' \otimes \varphi(m)]
\end{aligned}$$

is natural in M and N. For $M = R$ it is an isomorphism. Now note that the functors $\widehat{R} \otimes_R \operatorname{Hom}_R(-, N)$ and $\operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R -, \widehat{R} \otimes_R N)$ from R-mod$^{\mathrm{op}}$ to $\widehat{R}$-mod are both left exact for any N. Hence the above map is a natural isomorphism on R-mod.

We have an embedding of R-modules $R \hookrightarrow \widehat{R}$. Note that $C \cong \widehat{R} \otimes_R C$ by our choice of I.

Applying $\operatorname{Hom}_R(C, -)$ to the epimorphism $B \twoheadrightarrow C$ we obtain the first row of the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Hom}_R(C,B) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_R(C,C) \arrow[d, equal] \\
\widehat{R} \otimes_R \operatorname{Hom}_R(C,B) \arrow[r] \arrow[d, equal] & \widehat{R} \otimes_R \operatorname{Hom}_R(C,C) \arrow[d, equal] \\
\operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R C, \widehat{R} \otimes_R B) \arrow[r] & \operatorname{Hom}_{\widehat{R}}(\widehat{R} \otimes_R C, \widehat{R} \otimes_R C)
\end{tikzcd}$$

By assumption the morphism in the first row is not onto, hence neither is the morphism in the third row. This means that the short exact sequence of $\widehat{R}$-modules

$$\widehat{R} \otimes_R A \rightarrowtail \widehat{R} \otimes_R B \twoheadrightarrow \widehat{R} \otimes_R C$$

is non-split.

We want to show that this sequence, together with the $\widehat{R}$ finitely generated projective $S \otimes_k \widehat{R}$-module $L \otimes_R \widehat{R}$, satisfies the assumptions of Lemma 7.2. It only remains to see that $L \otimes_R \widehat{R}$ induces a representation embedding. The finite dimensional $\widehat{R}$-modules are exactly the finite dimensional R-modules which are annihilated by some power of I. For such a module M we clearly have $L \otimes_R M \cong L \otimes_R \widehat{R} \otimes_{\widehat{R}} M$, so $L \otimes_R \widehat{R}$ induces a representation embedding since L induces a representation embedding by assumption.

So, by Lemma 7.2, we know that the last row of the following commutative diagram does not split.

$$\begin{tikzcd}
L \otimes_R A \arrow[r, tail] \arrow[d] & L \otimes_R B \arrow[r, two heads] \arrow[d] & L \otimes_R C \arrow[d, equal] \\
L \otimes_R \widehat{R} \otimes_R A \arrow[r, tail] \arrow[d, equal] & L \otimes_R \widehat{R} \otimes_R B \arrow[r, two heads] \arrow[d, equal] & L \otimes_R C \arrow[d, equal] \\
(L \otimes_R \widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_R A) \arrow[r, tail] & (L \otimes_R \widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_R B) \arrow[r, two heads] & (L \otimes_R \widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_R C)
\end{tikzcd}$$

Since it is a pushout of the first row, this row does not split either. □

7.4. **Lemma.** *Theorem 7.1 holds if* $\dim_k C < \infty$.

Proof. Assume the sequence $L \otimes_R A \rightarrowtail L \otimes_R B \twoheadrightarrow L \otimes_R C$ splits. Let $h : L \otimes_R C \longrightarrow L \otimes_R B$ be a splitting. Note that $\dim_k L \otimes_R C < \infty$. Let $\{c_i, \dots c_r\}$ be a k-basis of $L \otimes_R C$. We write $h(c_i) = \sum_{j=1}^{N_i} l_{i,j} \otimes b_{i,j}$ with $l_{i,j} \in L$ and $b_{i,j} \in B$. We denote by B_0 the finitely generated submodule of B generated by the $b_{i,j}$. Then by definition h factors through $L \otimes_R B_0 \hookrightarrow L \otimes_R B$. We obtain the following diagram, where A_0 is the kernel of the map $B_0 \twoheadrightarrow C$ and the map $A_0 \rightarrowtail A$ is the kernel morphism.

$$\begin{tikzcd}
A_0 \arrow[r, tail] \arrow[d, tail] & B_0 \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, equal] \\
A \arrow[r, tail] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Since the lower sequence is the pushout of the upper sequence, the upper sequence is also non split. However we have just seen that the upper sequence splits when tensored with L. This contradicts Lemma 7.2 for R complete, and Lemma 7.3 for R commutative. □

7.5. **Lemma.** *Theorem 7.1 holds if* $\dim_k A < \infty$.

Proof. We denote by $-^*$ the functor $\operatorname{Hom}(-, k) : R\text{-Mod}^{op} \rightleftarrows R\text{-Mod}$. We obtain the following commutative diagram of R-modules, where the vertical maps are the natural embedding of the spaces into their double duals.

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, tail] \\
A^{**} \arrow[r, tail] & B^{**} \arrow[r, two heads] & C^{**}
\end{tikzcd}$$

Since the original sequence is a pullback of its double dual, the double dual cannot split. Hence also the dual sequence

$$C^* \rightarrowtail B^* \twoheadrightarrow A^*$$

is non-split. We will show that it fulfills the assumptions of Lemma 7.4 together with the R finitely generated projective $S^{\text{op}} \otimes_k R$-lattice $\widetilde{L} = \operatorname{Hom}_R(L, R)$. Indeed we have

$$\begin{aligned}
\widetilde{L} \otimes_{R^{\text{op}}} M^* &= \operatorname{Hom}_R(L, R) \otimes_{R^{\text{op}}} M^* \\
&= \operatorname{Hom}_R(L, M^*) \\
&= \operatorname{Hom}_R(L, \operatorname{Hom}_k(M, k)) \\
&= \operatorname{Hom}_k(L \otimes_R M, k) \\
&= (L \otimes_R M)^*
\end{aligned}$$

Since a finite dimensional module M is indecomposable if and only if M^* is so, and $L \otimes_R M$ is indecomposable if and only if $(L \otimes_R M)^*$ is so, the fact that $\widetilde{L} \otimes_{R^{\text{op}}} -$ preserves indecomposables follows from the fact that $L \otimes_R -$ preserves indecomposables. By the same argument one sees that $\widetilde{L} \otimes_{R^{\mathrm{op}}} -$ reflects isomorphism classes since $L \otimes_R -$ reflects isomorphism classes.

Hence by Lemma 7.4 the sequence

$$\begin{tikzcd} \widetilde{L} \otimes_{R^{\mathrm{op}}} C^* \arrow[r, tail] \arrow[d, equal] & \widetilde{L} \otimes_{R^{\mathrm{op}}} B^* \arrow[r, two heads] \arrow[d, equal] & \widetilde{L} \otimes_{R^{\mathrm{op}}} A^* \arrow[d, equal] \\ (L \otimes_R C)^* \arrow[r, tail] & (L \otimes_R B)^* \arrow[r, two heads] & (L \otimes_R A)^* \end{tikzcd}$$

is non-split. But since this is the dual of the sequence

$$L \otimes_R A \rightarrowtail L \otimes_R B \twoheadrightarrow L \otimes_R C$$

this sequence has to also be non-split. $\square$

8. Proof of Theorem 4.3

For $(\alpha, \beta) \in k^2$ we denote the $f_{\alpha\beta}$ corresponding to the submodule L^i of L by $f^i_{\alpha\beta}$. We set

$$H_i = \{(\alpha, \beta) \in k^2 \mid f^i_{\alpha\beta} \text{ factors through } \pi_{\alpha\beta}\}.$$

The proof of Theorem 4.3 now relies on the following two Lemmas.

8.1. **Lemma.**

$$\bigcap_{i=1}^{\infty} H_i = \emptyset$$

8.2. **Lemma.** *For any i the set H_i is Zariski-constructible in k^2.*

Proof of Theorem 4.3. Assume all the H_i are dense in k^2. Since H_i is constructible by Lemma 8.2 it contains a non-empty subset $U_i \subseteq H_i$ which is open in k^2. Now by Lemma 8.1 we have $\cap_{i=1}^{\infty} U_i = \emptyset$. Then, taking a basic open subset $D(f_i) \subseteq U_i$ for every $i > 0$, we see that $k^2 = \cup_{i=0}^{\infty} V(f_i)$ is a countable union of curves. This is impossible, since every line intersects a curve in only finitely many points, but our field is uncountable. Hence there is some i such that the set H_i is not dense. $\square$

Proof of Lemma 8.1. Let $(\alpha, \beta) \in k^2$. We denote the $f_{\alpha\beta}$ corresponding to L itself by $f^{\infty}_{\alpha\beta}$. Note that the pullback

$$k[X,Y]/(X-\alpha, Y-\beta) \rightarrowtail E \twoheadrightarrow (X-\alpha, Y-\beta)$$

of the short exact sequence

$$\frac{k[X,Y]}{(X-\alpha, Y-\beta)} \xrightarrow{\cdot(X-\alpha)} \frac{k[X,Y]}{((X-\alpha)^2, Y-\beta)} \twoheadrightarrow \frac{k[X,Y]}{(X-\alpha, Y-\beta)}$$

along $L(X-\alpha, Y-\beta) \xrightarrow{\substack{(X-\alpha)\mapsto 0 \\ (Y-\beta)\mapsto 1}} k[X,Y]/(X-\alpha, Y-\beta)$ does not split. Tensoring with L we obtain the pullback depicted in the following diagram (tensoring with L is exact since L is projective over $k[X,Y]$).

$$\begin{tikzcd}
\frac{L}{(X-\alpha,Y-\beta)} \arrow[r, tail] \arrow[d, equal] & \frac{L}{((X-\alpha)^2,Y-\beta)} \arrow[r, two heads, "\pi_{\alpha\beta}"] \arrow[dr, phantom, "\mathrm{PB}"] & \frac{L}{(X-\alpha,Y-\beta)} \\
\frac{L}{(X-\alpha,Y-\beta)} \arrow[r, tail] & L\otimes_{k[X,Y]} E \arrow[r, two heads] \arrow[u] & L(X-\alpha,Y-\beta) \arrow[u, "f^\infty_{\alpha\beta}"]
\end{tikzcd}$$

By Theorem 7.1 the lower sequence does not split, so $f^\infty_{\alpha\beta}$ does not factor through $\pi_{\alpha\beta}$. By Lemma 6.2

$$\begin{aligned}
&\operatorname{Ext}^1(L(X-\alpha,Y-\beta), \frac{L}{(X-\alpha,Y-\beta)}) \\
&\qquad = \varprojlim_i \operatorname{Ext}^1(L^i\cap L(X-\alpha,Y-\beta), \frac{L}{(X-\alpha,Y-\beta)}),
\end{aligned}$$

so there has to be $i_{\alpha\beta}\in\mathbb{N}$ such that $f^{i_{\alpha\beta}}_{\alpha\beta}$ does not factor through $\pi_{\alpha\beta}$. This shows that $(\alpha,\beta)\notin H_i$, so $(\alpha,\beta)\notin\cap_{i=1}^\infty H_i$. Since this argument works for every $(\alpha,\beta)\in k^2$ it follows that $\cap_{i=1}^\infty H_i=\emptyset$. □

For the proof of Lemma 8.2 we will need the following observation.

8.3. **Lemma.** *Let* $A\xrightarrow{\varphi} B \xleftarrow{\psi} C$ *in* $\Lambda\otimes_k k[X,Y]$-r. lat. *Then the set*

$$\{(\alpha,\beta)\in k^2 \mid \psi \underset{k[X,Y]}{\otimes} \tfrac{k[X,Y]}{(X-\alpha,Y-\beta)} \text{ factors through } \varphi \underset{k[X,Y]}{\otimes} \tfrac{k[X,Y]}{(X-\alpha,Y-\beta)}\}$$

is constructible.

Proof. We denote by $\mathcal{L}$ the set given in the lemma. By choosing $k[X,Y]$-bases of the three lattices we have (α,β) is in $\mathcal{L}$ if and only if a certain finite system of linear equations (over $k[X,Y]$) is solvable modulo $(X-\alpha,Y-\beta)$. However, the solvability of a system of linear equations can be checked by investigating if certain subdeterminants are zero or non-zero. Clearly all subdeterminants are polynomials, so the claim follows. □

Proof of Lemma 8.2. We need to find a generic version of $f^i_{\alpha\beta}$ and $\pi_{\alpha\beta}$. To do so, we set

$$\begin{tikzcd}
\frac{L\otimes_k k[A,B]\otimes_k k[X,Y]}{((X-A)^2,Y-B)} \arrow[r, two heads, "\pi"] & \frac{L\otimes_k k[A,B]\otimes_k k[X,Y]}{(X-A,Y-B)} \\
& (L^i\otimes k[A,B])\cap((L\otimes_k k[A,B])(X-A,Y-B)) \arrow[u, "f^i"]
\end{tikzcd}$$

where π is the canonical projection and f^i is the composition

$$\begin{tikzcd}
(L^i \otimes k[A,B]) \cap ((L \otimes_k k[A,B])(X-A, Y-B)) \arrow[d, hook, "\text{embedding}"] \\
(L \otimes_k k[A,B])(X-A, Y-B) \arrow[d, "1_L \otimes \substack{(X-A)\mapsto 0 \\ (Y-B)\mapsto 1}"] \\
\dfrac{L \otimes_k k[A,B] \otimes_k k[X,Y]}{(X-A, Y-B)}
\end{tikzcd}$$

Now clearly

$$\pi_{\alpha\beta} = \pi \underset{k[A,B]}{\otimes} \frac{k[A,B]}{(A-\alpha, B-\beta)} \quad \text{and} \qquad f^i_{\alpha\beta} = f^i \underset{k[A,B]}{\otimes} \frac{k[A,B]}{(A-\alpha, B-\beta)},$$

so we can apply Lemma 8.3 to complete the proof. $\square$

9. Examples

The first example gives no new result, but it illustrates the idea with only very little calculation.

9.1. **Example.** Let $\Lambda = k[\circ \overset{a}{\underset{c}{\overset{\underrightarrow{\ b\ }}{\longrightarrow}}} \circ]$, $L = k[X,Y] \overset{1}{\underset{Y}{\overset{\underrightarrow{\ X\ }}{\longrightarrow}}} k[X,Y]$. Let L' be the Λ-submodule generated by $\{(1,0),(X,0),(Y,0)\}$. That is $\mathrm{Pol}^1 \overset{1}{\underset{Y}{\overset{\underrightarrow{\ X\ }}{\longrightarrow}}} \mathrm{Pol}^2$, if Pol^i denotes the polynomials of degree at most i. Let $(\alpha,\beta) \in (k^{\mathrm{sep}})^2$. We have to find out whether there is a map h as indicated in the following diagram.

$$\begin{tikzcd}
\begin{bmatrix} k_{\alpha\beta} \oplus k_{\alpha\beta}(X-\alpha) \\ 1 \downarrow \ X \downarrow \ \downarrow \beta \\ k_{\alpha\beta} \oplus k_{\alpha\beta}(X-\alpha) \end{bmatrix} \arrow[r] & \begin{bmatrix} k_{\alpha\beta} \\ 1 \downarrow \ \alpha \downarrow \ \downarrow \beta \\ k_{\alpha\beta} \end{bmatrix} \\
& \begin{bmatrix} (X-\alpha)k_{\alpha\beta} + (Y-\beta)k_{\alpha\beta} \\ 1 \downarrow \ X \downarrow \ \downarrow Y \\ (X-\alpha)\,\mathrm{Pol}^1_{\alpha\beta} + (Y-\beta)\,\mathrm{Pol}^1_{\alpha\beta} \end{bmatrix} \arrow[ul, dashed, "h"] \arrow[u, "\substack{(X-\alpha)\mapsto 0 \\ (Y-\beta)\mapsto 1}"']
\end{tikzcd}$$

Assume such an h exists. It can be assumed to be $k_{\alpha\beta}$-linear. Let the upper component of h map $(X-\alpha)$ to $h_1(X-\alpha)$ and $(Y-\beta)$ to

$1 + h_2(X - \alpha)$. Then in the lower component $(X - \alpha)(Y - \beta)$ is on the one hand mapped to

$$\begin{aligned}(X-\alpha)(Y-\beta) &= X(Y-\beta) - \alpha(Y-\beta)\\ &= b(Y-\beta) - \alpha a(Y-\beta)\\ &\mapsto b(1+h_2(X-\alpha)) - \alpha a(1+h_2(X-\alpha))\\ &= \alpha + (X-\alpha) + h_2(X-\alpha) - \alpha(1+h_2(X-\alpha))\\ &= X-\alpha\end{aligned}$$

and on the other hand to

$$\begin{aligned}(X-\alpha)(Y-\beta) &= Y(X-\alpha) - \beta(X-\alpha)\\ &= c(X-\alpha) - \beta a(X-\alpha)\\ &\mapsto c(h_1(X-\alpha)) - \beta a(h_1(X-\alpha))\\ &= \beta h_1(X-\alpha) - \beta h_1(X-\alpha)\\ &= 0\end{aligned}$$

This gives a contradiction. Therefore

$$\operatorname{repdim} \Lambda[L'] = 4.$$

However the algebra $\Lambda[L']$ is just $k[\circ \overset{a}{\underset{c}{\xrightarrow{\;b\;}}} \circ \overset{a}{\underset{c}{\xrightarrow{\;b\;}}} \circ]/(ab - ba, ac - ca, bc - cb)$, which has already bean treated in [5].

The examples in Table 1 all work in quite a similar way as the one above, except that the calculation gets longer due to the size of the diagrams. In all of them we take Λ to be the quiver algebra kQ with Q the quiver given in the table. Then L' is such that the assumption of Theorem 4.2 is satisfied. Therefore $\operatorname{repdim} kQ[L'] \geq 4$ for all $kQ[L']$ in Table 1. In all examples Iyama's general upper bound for the representation dimension ([4], the combination of Theorems 2.2.2 and 2.5.1) shows that we actually have $\operatorname{repdim} kQ[L'] = 4$.

Table 1: Examples

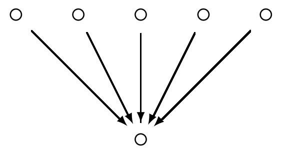
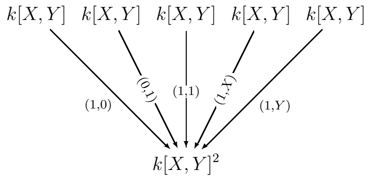
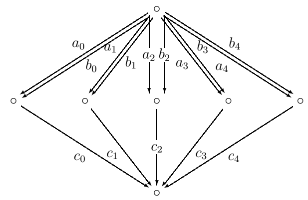
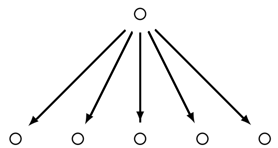
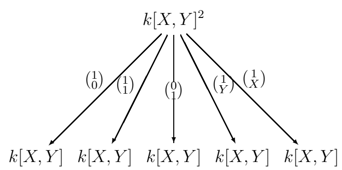
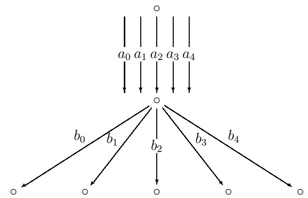
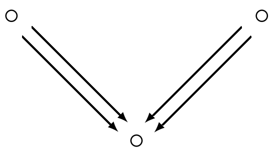
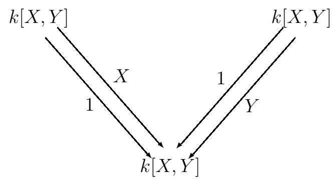
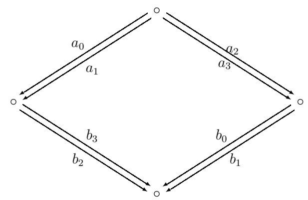
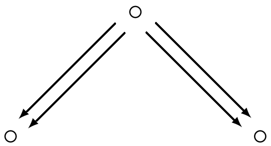
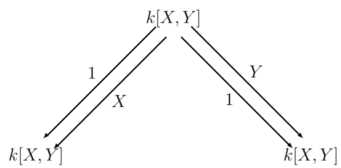
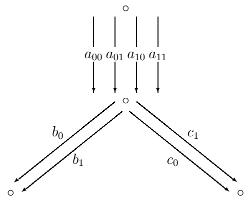
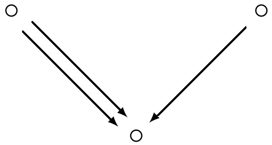
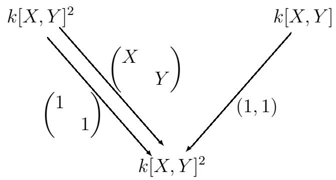
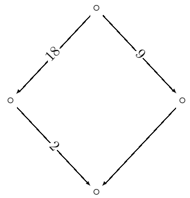
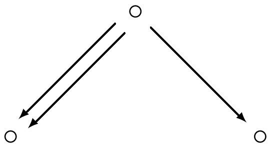
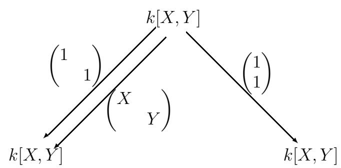
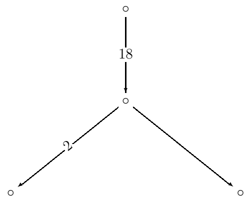
Q:	L:	Generators of L':	Quiver of $kQ[L']$:	Relations:
		$\begin{pmatrix} X-1 & 0 & 0 & 0 & 0 \\ & & 0 & & \end{pmatrix}$, $\begin{pmatrix} X-Y & 0 & 0 & 0 & 0 \\ & & 0 & & \end{pmatrix}$ $\begin{pmatrix} 0 & Y-X & 0 & 0 & 0 \\ & & 0 & & \end{pmatrix}$, $\begin{pmatrix} 0 & Y-XY & 0 & 0 & 0 \\ & & 0 & & \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & Y & 0 & 0 \\ & & 0 & & \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & Y-X & 0 & 0 \\ & & 0 & & \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 & Y-1 & 0 \\ & & 0 & & \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & Y & 0 \\ & & 0 & & \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 & 0 & X \\ & & 0 & & \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 & X-1 \\ & & 0 & & \end{pmatrix}$		$b_i c_i = a_{i-1} c_{i-1} + a_{i+1} c_{i+1}$ $i \in \mathbb{Z}/5\mathbb{Z}$
		$\begin{pmatrix} & (0,X-Y) & \\ 0 & \cdots & 0 \end{pmatrix}$ $\begin{pmatrix} & (-X,X) & \\ 0 & \cdots & 0 \end{pmatrix}$ $\begin{pmatrix} & (Y-X,0) & \\ 0 & \cdots & 0 \end{pmatrix}$ $\begin{pmatrix} & (XY,-X) & \\ 0 & \cdots & 0 \end{pmatrix}$ $\begin{pmatrix} & (X,-1) & \\ 0 & \cdots & 0 \end{pmatrix}$		$a_i b_i$, $(a_{i-1} + a_{i+1}) b_i$ $i \in \mathbb{Z}/5\mathbb{Z}$
		$\begin{pmatrix} 1 & & 0 \\ & 0 & \end{pmatrix}$, $\begin{pmatrix} Y & & 0 \\ & 0 & \end{pmatrix}$ $\begin{pmatrix} 0 & & 1 \\ & 0 & \end{pmatrix}$, $\begin{pmatrix} 0 & & X \\ & 0 & \end{pmatrix}$		$a_i b_j = a_j b_i$ $i, j \in \{0 \ldots 3\}$

Table 1: (continued)

<table>
<tr><th>Q:</th><th>L:</th><th>Generators of L':</th><th>Quiver of $kQ[L']$:</th><th>Relations:</th></tr>
<tr>
<td></td>
<td>
</td>
<td>$\left(\begin{smallmatrix} & p & \\ 0 & & 0\end{smallmatrix}\right)$
$p \in \{1, X, Y, XY\}$</td>
<td>
</td>
<td>$a_{00}b_1 = a_{10}b_0$
$a_{01}b_1 = a_{11}b_0$
$a_{00}c_1 = a_{01}c_0$
$a_{10}c_1 = a_{11}c_0$</td>
</tr>
<tr>
<td></td>
<td>
</td>
<td>$\left(\begin{smallmatrix}(p,0) & & 0\\ & 0 & \end{smallmatrix}\right)$, $\left(\begin{smallmatrix}(0,p) & & 0\\ & 0 & \end{smallmatrix}\right)$,
$\left(\begin{smallmatrix}(0,0) & & p\\ & 0 & \end{smallmatrix}\right)$
$p = X^iY^j$, $0 \leq i, j \leq 2$</td>
<td>
</td>
<td></td>
</tr>
<tr>
<td></td>
<td>
</td>
<td>$\left(\begin{smallmatrix} & (p,0) & \\ 0 & & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & (0,p) & \\ 0 & & 0\end{smallmatrix}\right)$,
$p = X^iY^j$, $0 \leq i, j \leq 2$</td>
<td>
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</table>

10. Construction of n-point extensions of large representation dimension

Corollary 4.4 says that any wild algebra has a one-point extension of representation dimension at least four, provided the base-field is sufficiently large. We wish to remove this assumption on the field. In fact we will prove the following more general result

10.1. **Theorem.** *Let Λ be a finite dimensional wild algebra, $n \in \mathbb{N}$. Then there is an iterated one-point extension $\Lambda[M_1] \cdots [M_n]$ ($M_i \in \Lambda[M_1] \cdots [M_{i-1}]$-mod — such an iterated one-point extension will be called n-point extension) such that*

$$\operatorname{repdim} \Lambda[M_1] \cdots [M_n] \geq n+3.$$

For the proof we will use the fact that we know the theorem holds for some algebras (which we do from [5]). Then we use a representation embedding to carry it over to an arbitrary wild algebra.

We will now be using the polynomial ring $k[\mathbf{X}] = k[X_0, \ldots, X_n]$.

10.2. **Setup.** Assume we have finite dimensional algebras Λ_0, Γ and an extension $\Sigma_0 = \left(\begin{smallmatrix} \Gamma & \\ M_0 & \Lambda_0 \end{smallmatrix}\right)$. Moreover we assume there is a $\Sigma_0 \otimes_k k[\mathbf{X}]$-lattice $L_0 = \left(\begin{smallmatrix} L_\Gamma \\ L_{\Lambda_0} \end{smallmatrix}\right)$ such that

(1) the algebra Γ is triangular with exactly n simple modules (up to isomorphism) and all of these have trivial endomorphism ring (this means there are simples $S_1, \ldots, S_n$ such that $\forall i$: $\operatorname{End}_\Gamma(S_i) = k$ and $\forall i \leq j : \operatorname{Ext}^1_\Gamma(S_j, S_i) = 0$ – hence extensions with Γ are n-point extensions),

(2) there is a non-empty open subset $U \subseteq \operatorname{MaxSpec} k[\mathbf{X}]$ such that for any $\mathfrak{p} \in U$ the first n terms of the projective resolution of $L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p}$ have the form

$$\begin{pmatrix} P_{n-1} \\ M_0 \otimes_\Gamma P_{n-1} \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} P_0 \\ M_0 \otimes_\Gamma P_0 \end{pmatrix} \twoheadrightarrow L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p},$$

for projective Γ-modules $P_0, \ldots P_{n-1}$,

(3) the lattice L_0 satisfies the assumption of Theorem 3.5 for $d = n+1$. In particular $\operatorname{repdim} \Lambda_0 \geq n+3$.

10.3. **Remark.** It is shown implicitly in [5], Examples 7.2 and 7.3 that the following two sets of algebras fulfill all the assumptions of Setup 10.2. We denote by $Q_{N,L}$ the quiver

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r, shift left=2, "x_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "x_N"'] & \overset{2}{\circ} \arrow[r, shift left=2, "x_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "x_N"'] & \overset{3}{\circ} & \cdots & \overset{L-1}{\circ} \arrow[r, shift left=2, "x_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "x_N"'] & \overset{L}{\circ} \end{tikzcd}.$$

(1) – from [5, 7.2]:

$$\begin{aligned}\Lambda_0 &= kQ_{n+2,2}\\ \Gamma &= kQ_{n+2,n}/(x_ix_j + x_jx_i, x_i^2 \mid 1 \leq i,j \leq n+2)\\ \Sigma &= kQ_{n+2,n+2}/(x_ix_j + x_jx_i, x_i^2 \mid 1 \leq i,j \leq n+2)\end{aligned}$$

(2) – from [5, 7.3]:

$$\begin{aligned}\Lambda_0 &= kQ_{n+2,2}\\ \Gamma &= kQ_{n+2,n}/(x_ix_j - x_jx_i \mid 1 \leq i,j \leq n+2)\\ \Sigma &= kQ_{n+2,n+2}/(x_ix_j - x_jx_i \mid 1 \leq i,j \leq n+2)\end{aligned}$$

The important point for us is: It is possible to find algebras satisfying the assumptions of Setup 10.2.

10.4. **Proposition.** *With the setup above, let Λ be another algebra such that there is a representation embedding $\mathtt{F} : \Lambda_0\text{-mod} \longrightarrow \Lambda\text{-mod}$. Then for $M = \mathtt{F}M_0 \in \Lambda \otimes_k \Gamma^{\mathrm{op}}\text{-mod}$ we have*

$$\operatorname{repdim} \begin{pmatrix} \Gamma & \\ M & \Lambda \end{pmatrix} \geq n+3.$$

Note that Theorem 10.1 follows immediately from this proposition.

Proof of Proposition 10.4. We set $\Sigma = \left(\begin{smallmatrix} \Gamma & \\ M & \Lambda \end{smallmatrix}\right)$. Then $\mathtt{F}$ extends to a functor $\Sigma_0\text{-Mod} \longrightarrow \Sigma\text{-Mod}$ by

$$\mathtt{F}(\begin{pmatrix} X_\Gamma \\ X_{\Lambda_0} \end{pmatrix}_\varphi) = \begin{pmatrix} X_\Gamma \\ \mathtt{F}X_{\Lambda_0} \end{pmatrix}_{\mathtt{F}\varphi}.$$

Note that for a projective Γ-module P, the projective Σ_0-module $\left(\begin{smallmatrix} P \\ M_0 \otimes_\Gamma P \end{smallmatrix}\right)$ is mapped to the projective Σ-module $\left(\begin{smallmatrix} P \\ M \otimes_\Gamma P \end{smallmatrix}\right)$.

To show that the representation dimension of Σ is at least $n+3$, we apply Theorem 3.5 with the lattice $L = \mathtt{F}L_0$ and $d = n+1$.

Assume $\mathfrak{p}$ is in the open set U described in Setup 10.2(2), and let

$$\mathbb{E} : k[\mathbf{X}]/\mathfrak{p} \rightarrowtail E_n \longrightarrow \cdots \longrightarrow E_0 \longrightarrow k[\mathbf{X}]/\mathfrak{p}$$

be an $n+1$ extension of $k[\mathbf{X}]$-modules. To find out whether the $n+1$-extension $L_0 \otimes_{k[\mathbf{X}]} \mathbb{E}$ splits we compare it to a projective resolution of $L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p}$ as indicated in the following diagram.

$$\begin{tikzcd}[column sep=small]
L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p} \arrow[r, tail] & L_0 \otimes_{k[\mathbf{X}]} E_n \arrow[r] & L_0 \otimes_{k[\mathbf{X}]} E_{n-1} \arrow[r] & \cdots \arrow[r] & L_0 \otimes_{k[\mathbf{X}]} E_0 \arrow[r] & L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p} \\
L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p} \arrow[u, equal] \arrow[r, tail] & \mathrm{PB} \arrow[u] \arrow[r] \arrow[dr] & \begin{pmatrix} P_{n-1} \\ M_0 \otimes_\Gamma P_{n-1} \end{pmatrix} \arrow[u] \arrow[r] & \cdots \arrow[r] & \begin{pmatrix} P_0 \\ M_0 \otimes_\Gamma P_0 \end{pmatrix} \arrow[u] \arrow[r] & L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p} \arrow[u, equal] \\
& & \Omega^n(L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p}) \arrow[u] & & &
\end{tikzcd}$$

We denote the short exact sequence $L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p} \rightarrowtail \mathrm{PB} \twoheadrightarrow \Omega^n(L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p})$ in the diagram above by $\widehat{\mathbb{F}}$. Note that by our assumptions on Γ the module $\Omega^n(L_0 \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p})$ is of the form $\left(\begin{smallmatrix} 0 \\ H \end{smallmatrix}\right)$ for some Λ_0-module

H. We denote by $\mathbb{F}$ the one-extension between the Λ_0-modules H and $L_{\Lambda_0} \otimes_{k[\mathbf{X}]} k[\mathbf{X}]/\mathfrak{p}$ in the second component of $\widehat{\mathbb{F}}$. Now observe that

$$\begin{aligned}
& L_0 \otimes_{k[\mathbf{X}]} \mathbb{E} \text{ splits as } n+1\text{-extension of } \Sigma_0\text{-modules} \\
\iff & \widehat{\mathbb{F}} \text{ splits} \\
\iff & \mathbb{F} \text{ splits} \\
\iff & \mathtt{F}\mathbb{F} \text{ splits}
\end{aligned}$$

since $\mathtt{F}$ is a representation embedding, and by applying $\mathtt{F}$ to the diagram above

$$\begin{aligned}
\iff & \mathtt{F}\widehat{\mathbb{F}} \text{ splits} \\
\iff & \mathtt{F}\mathbb{E} \text{ splits as } n+1\text{-extension of } \Sigma\text{-modules.}
\end{aligned}$$

By 3 of Setup 10.2, the set

$$\begin{aligned}
V = \{\mathfrak{p} \in \operatorname{MaxSpec} k[\mathbf{X}] \mid \exists \mathbb{E} \in \operatorname{Ext}^{n+1}_{k[\mathbf{X}]}(k[\mathbf{X}]/\mathfrak{p}, k[\mathbf{X}]/\mathfrak{p}) \text{ such that} \\
L_0 \otimes_{k[\mathbf{X}]} \mathbb{E} \text{ is non-split as } n+1\text{-extension of } \Sigma_0\text{-modules}\}
\end{aligned}$$

is dense in $\operatorname{MaxSpec} k[\mathbf{X}]$. Therefore so is its intersection $U \cap V$ with the open set from 10.2(2). By the equivalences above we have that $U \cap V$ is contained in the set

$$\begin{aligned}
\{\mathfrak{p} \in \operatorname{MaxSpec} k[\mathbf{X}] \mid \exists \mathbb{E} \in \operatorname{Ext}^{n+1}_{k[\mathbf{X}]}(k[\mathbf{X}]/\mathfrak{p}, k[\mathbf{X}]/\mathfrak{p}) \text{ such that} \\
L \otimes_{k[\mathbf{X}]} \mathbb{E} \text{ is non-split as } n+1\text{-extension of } \Sigma\text{-modules}\}.
\end{aligned}$$

Hence we may apply Theorem 3.5, and obtain $\operatorname{repdim} \Sigma \geq n+3$. $\square$

10.5. **Corollary.** *For any $n \in \mathbb{N}$ and $i \in \{1,2,3\}$ there is $a_n \in \mathbb{N}$ such that there is an algebra of representation dimension $n+3$ with quiver Q_i as below.*

$$Q_1 \quad \circ \xrightarrow{3} \circ \xrightarrow{a_n} \underset{1}{\circ} \xrightarrow{n+2} \underset{2}{\circ} \xrightarrow{n+2} \underset{3}{\circ} \quad \cdots \quad \underset{n-1}{\circ} \xrightarrow{n+2} \underset{n}{\circ}$$

$$Q_2 \quad \begin{tikzcd}
\circ \arrow[dr, "2"] \\
& \circ \arrow[r, "a_n"] & \underset{1}{\circ} \arrow[r, "n+2"] & \underset{2}{\circ} \arrow[r, "n+2"] & \underset{3}{\circ} & \cdots & \underset{n-1}{\circ} \arrow[r, "n+2"] & \underset{n}{\circ} \\
\circ \arrow[ur]
\end{tikzcd}$$

$$Q_2 \quad \begin{tikzcd}
\circ \arrow[ddr] \\
\circ \arrow[dr] \\
\circ \arrow[r] & \circ \arrow[r, "a_n"] & \underset{1}{\circ} \arrow[r, "n+2"] & \underset{2}{\circ} \arrow[r, "n+2"] & \underset{3}{\circ} & \cdots & \underset{n-1}{\circ} \arrow[r, "n+2"] & \underset{n}{\circ} \\
\circ \arrow[ur] \\
\circ \arrow[uur]
\end{tikzcd}$$

(an arrow $\xrightarrow{n}$ stand for n arrows in that position)

Proof. In the setup of Proposition 10.4 we choose Λ_0, Γ, and M_0 as in any of the two examples presented in Remark 10.3. We choose $\Lambda = k\left[\begin{tikzcd}[cramped] \circ \arrow[r, shift left=1] \arrow[r] \arrow[r, shift right=1] & \circ \end{tikzcd}\right], k\left[\begin{tikzcd}[cramped] \circ \arrow[dr, shift left=1] \arrow[dr, shift right=1] & \\ & \circ \\ \circ \arrow[ur] & \end{tikzcd}\right]$, and $k\left[\begin{tikzcd}[cramped, row sep=tiny] \circ \arrow[ddr] & \\ \circ \arrow[dr] & \\ \circ \arrow[r] & \circ \\ \circ \arrow[ur] & \\ \circ \arrow[uur] & \end{tikzcd}\right]$, respectively for the three cases of the corollary. Since these algebras are wild, a representation embedding as required by Proposition 10.4 exists. $\square$

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